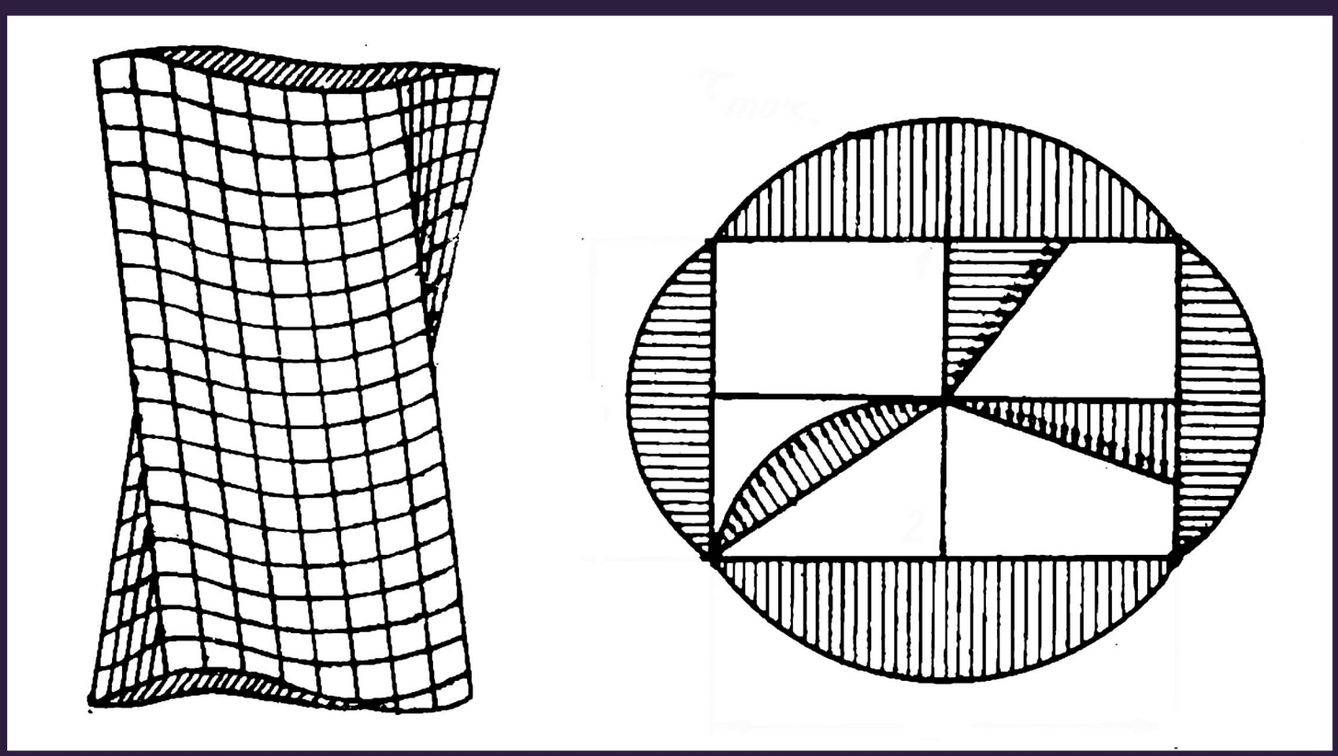
P. STEPIN

STRENGTH of MATERIALS

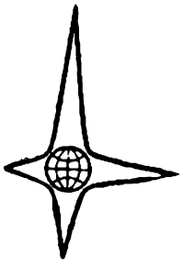


TO THE READER

Peace Publishers would be glad to have your opinion of the translation and the design of this book.

Please send all your suggestions to 2, Pervy Rizhsky Pereulok, Moscow, U.S.S.R.

Printed in the Union of Soviet Socialist Republics



PEACE
PUBLISHERS

П. А. СТЕПИН

СОПРОТИВЛЕНИЕ МАТЕРИАЛОВ

Издание второе

ИЗДАТЕЛЬСТВО «ВЫСШАЯ ШКОЛА»
МОСКВА

На английском языке

P. STEPIN

STRENGTH OF MATERIALS

Second Edition
Revised and Enlarged

TRANSLATED FROM THE RUSSIAN
BY
MARIA KONYAEVA

PEACE PUBLISHERS
MOSCOW

UDC 620.10(075.8)=20

PUBLISHER'S NOTE

The book is intended as a textbook on Strength of Materials for students at technical institutes.

It deals with tension and compression, torsion, bending, complex resistance of materials. It presents strength theories, stability analysis, dynamic problems, fatigue analysis.

At the end of the book the author discusses briefly the limit-design method and the principal ideas of the statistical method of structural design.

The book includes numerous examples and their solutions which facilitate the understanding of the subject-matter and clarify the essence of mechanical phenomena.

The book is very popular among students and professors at the Soviet technical institutes. This is a revised edition in English.

CONTENTS

CHAPTER I. FUNDAMENTALS

1. Problems Dealt with in the Course 9
2. Assumptions in Strength of Materials 13
3. Systems of Mechanical Units 16
4. External Forces (Loads) 18
5. Deformations and Displacements 22
6. Method of Sections 23
7. Stresses 26

CHAPTER II. TENSION AND COMPRESSION

8. Determination of Internal Forces 32
9. Determination of Stresses 33
10. Determination of Deformations and Displacements 34
11. Experimental Study of the Properties of Materials 41
12. Factor of Safety. Choice of Allowable Stresses 61
13. Basic Types of Problems in the Design of Stretched (Compressed) Bars 64
14. Statically Indeterminate Problems in Tension and Compression 68
15. Thermal and Assembly Stresses. Initial Control of Forces in Structures 73
16. Stresses on Inclined Sections under Tension (Compression) in One Direction 79
17. Law of Conjugate Shearing Stresses 81
18. Determination of Stresses on Inclined Sections under Tension (Compression) in Two Directions 82
19. Determination of Principal Stresses and Position of Principal Areas 84
20. Relation Between Strains and Stresses in Plane and Three-Dimensional States of Stress (Generalised Hooke's Law) 87
21. Work of External and Internal Forces in Tension (Compression). Potential Energy of Strain 89
22. Stress Concentration. Contact Stresses 94

CHAPTER III. SHEAR

23. Determination of Stresses 100
24. Deformations in Shear 106
25. Potential Energy in Shear. Relation Between Three Elastic Constants E, G and μ 107

CHAPTER IV. GEOMETRICAL CHARACTERISTICS OF SECTIONS

26. Statical Moment of an Area 111
27. Moment of Inertia of an Area 113
28. Relation Between Moments of Inertia with Respect to Parallel Axes 115
29. Moments of Inertia of Simple Sections 116
30. Moments of Inertia of Composite Figures 118
31. Change of Moments of Inertia with Rotation of Axes 119
32. Principal Axes of Inertia and Principal Moments of Inertia 120
33. Relation Between Products of Inertia with Respect to Two Sets of Parallel Axes 122

CHAPTER V. TORSION

34. Construction of Twisting Moment Diagrams 130
35. Determination of Stresses in Bars of Circular Cross Section 133
36. Relation Between Moment Transmitted by the Shaft, Power and Angular Velocity 139
37. Deformations of Circular Bars in Torsion 140
38. Construction of Diagrams for Angles of Twist 142
39. Potential Energy in Torsion 144
40. Basic Results of Theory of Torsion of Non-Circular Bars . . 146
41. Torsion of Thin-Walled Bars of Closed Section 149
42. Statically Indeterminate Problems 152
43. Stress Concentration 155
44. Rational Shapes of Sections in Torsion 156

CHAPTER VI. BENDING. DETERMINATION OF STRESSES

45. Basic Concepts in Bending 160
46. Types of Supports of Beams 161
47. Determination of Reactions of Supports 162
48. Determination of Internal Forces 164
49. Sign Rule for Bending Moments and Shearing Forces 165
50. Relation Between Bending Moment, Shearing Force and Intensity of Distributed Load 166
51. Construction of Bending Moment and Shearing Force Diagrams 168
52. Determination of Normal Stresses 178
53. Condition of Strength Based on Normal Stresses 183
54. Determination of Shearing Stresses 186
55. Stresses on Inclined Sections of a Beam. Principal Stresses 194

56. Stress Concentration in Bending 195
57. Potential Energy of Strain in Bending 197

CHAPTER VII. BENDING. DETERMINATION OF DISPLACEMENTS

58. Differential Equation of Elastic Curve 199
59. Determination of Displacements When There Are Several Regions of Loading. Universal Formula 203
60. Examples of Determining Displacements in Bending by Use of the Universal Formula 207
61. Reciprocal Theorem 212
62. Mohr's Method of Determining Displacements 216
63. Analysis of Statically Indeterminate Beams 231
64. Examples of Calculating Statically Indeterminate Beams . . 234
65. Principles of General Method of Analysis of Statically Indeterminate Systems (Principles of Force Method) 238
66. Rational Arrangement of Supports of Beams 246
67. Rational Shapes of Cross Sections of Beams 250

CHAPTER VIII. STRENGTH THEORIES

68. Object of Strength Theories 257
69. First Theory of Strength 263
70. Second Theory of Strength 265
71. Third Theory of Strength 266
72. Energy Theories of Strength 268
73. Trends in Development of Strength Theories 269

CHAPTER IX. COMPLEX DEFORMATIONS (COMPLEX RESISTANCE)

74. Basic Concepts 274
75. Examples of Constructing Internal Force Diagrams for Bar with Broken Axis 275
76. Bending in Two Planes (Oblique Bending) 277
77. Bending with Tension (Compression) 285
78. Eccentric Compression (Tension) 287
79. Core of a Cross Section 292
80. Torsion and Shear. Design of Closely Coiled Springs . . . 293
81. Torsion with Bending 297
82. Torsion with Tension (Compression) 300
83. An Example of Design of a Shaft Subjected to Bending with Torsion 303
84. Analysis of Thin-Walled Vessels 305

CHAPTER X. STABILITY ANALYSIS OF COMPRESSED MEMBERS (BUCKLING)

85. Stable and Unstable Configurations of Equilibrium 311
86. Euler's Formula for the Critical Force 313

87. Effect of End Conditions of a Column 315
88. Limits of Applicability of Euler's Formula 316
89. Empirical Formulas for Determining Critical Stresses . . . 318
90. Practical Formula for Stability Analysis 318
91. Rational Shapes of Sections of Compression Members . . . 321
92. Bending Induced by Axial Loading 324

CHAPTER XI. STRENGTH ANALYSIS UNDER DYNAMIC LOADS

93. Dynamic Loads 330
94. Calculation of Stresses in Uniformly Accelerated Motion 331
95. Strength Analysis under Impact 333
96. Testing of Materials under Impact Loading (Impact Test) 336
97. Free Vibrations of a System with One Degree of Freedom 339
98. Forced Vibrations of a System with One Degree of Freedom. Resonance 342

CHAPTER XII. STRENGTH ANALYSIS UNDER ALTERNATING STRESSES (FATIGUE ANALYSIS)

99. Basic Definitions 345
100. Fatigue Curve for Complete Reversals of Stresses. Endurance Limit 349
101. Endurance Limit Diagram 351
102. Factors Affecting the Endurance Limit 355
103. Determination of Safety Factors for Completely Reversed Cycles of Stress 359
104. Determination of Safety Factor for Any Cycle of Varying Stress 360
105. Practical Measures to Combat Fatigue of Materials 363

CHAPTER XIII. RESISTANCE OF MATERIALS TO PLASTIC DEFORMATIONS. LIMIT DESIGN

106. Design Diagrams for an Elastic-Plastic Material 366
107. Plastic Design of Statically Indeterminate Systems Acting in Tension-Compression 368
108. Plastic Torsion of a Circular Bar 371
109. Plastic Bending of Statically Determinate Beams 373
110. Yield Conditions 375
111. Plastic Torsion with Tension of a Circular Bar 376
112. Plastic Bending with Tension and Shear 377
113. The Basic Principles of Limit Design 377
114. Conclusion. Current Trends in Development of Methods of Structural Design 380

APPENDICES 384

Index 396

CHAPTER I

FUNDAMENTALS

1. Problems Dealt with in the Course

Various structures and machines with the design and construction of which the engineer is concerned in his practical work must necessarily possess, apart from other qualities, *strength*, i.e., the ability to resist failure when subjected to external forces (loads).

This demands that the elements (parts) of structures and machines should be made of appropriate material and have suitable dimensions.

The treatment of methods of structural design based on strength constitutes the first problem of the course.

In addition, in many cases it is necessary to determine the changes in shape and dimensions (deformations) which are produced by loads in structural elements.

The fact is that in reality there exist no absolutely rigid, non-deformable bodies which are studied in theoretical mechanics.

Of course, deformations produced by ordinary working loads are small and can be detected only with special instruments.

Small deformations do not appreciably affect the laws of equilibrium and motion of a body and are consequently neglected in theoretical mechanics. However, without a study of these deformations it is difficult to solve a problem which is very important to the engineer, that is, to discover under what conditions failure may occur and, on the other hand, what are the safe conditions.

Moreover, in many cases it is necessary to limit the magnitude of deformations in spite of their smallness as compared with the dimensions of the part itself because otherwise the normal service of the structure may become impossible. For example, when a part is machined

deformation of the part itself and of the machine elements may lead to a reduction in accuracy beyond what is permissible.

The ability of a structural element to resist deformation is called *stiffness.*

Hence the second problem of the course: the treatment of methods of structural design based on stiffness.

The next problem of strength of materials is concerned with the study of stability of equilibrium forms of real (i.e., deformable) bodies.

Stability is the ability of a body to retain under loading its *original* state of equilibrium.

Instability is a condition in which the load causes an abrupt qualitative change in the shape of a part and the character of its deformation.

A thin member compressed by a force acting along its axis is an example of instability. Up to a certain specific value of the compressive force depending on the material, dimensions and supporting conditions of the member, it will remain straight. With an increase in the load, however, the member will buckle and become curved.

Instability may occur at magnitudes of loads that are quite safe from the viewpoint of the strength or stiffness of a structural element.

Methods of structural design based on stability constitute the third problem of the course.

In carrying out the above types of design it is necessary to aim at the maximum *economy in materials*, at adequate but not excessive dimensions of machine and structural parts. Obviously this requires the most complete and profound study of the properties of the materials used and the nature of the loads acting upon the part in question. This can be achieved by thorough experimental research and a careful study of experience accumulated in the designing of structures and the service given by them.

On the other hand, in the derivation of basic design relations in strength of materials it is necessary to introduce various hypotheses and simplifying assumptions. The validity of these hypotheses and assumptions, and also the degree of inaccuracy they lead to in design formulas are verified by comparing the results calculated on the basis of these formulas with experimental data.

Most of the structures an engineer has to deal with are very complicated in form, but their individual elements can be reduced to the following simplest types.

1. A *rod* is a body two dimensions of which are small as compared with the third (Fig. 1.1*a*). In a particular case a rod may have a constant cross-sectional area and a straight-line axis.

The axis of a rod is a line passing through the centroids of its cross sections.

A rod with a straight-line axis is often called a *bar.*

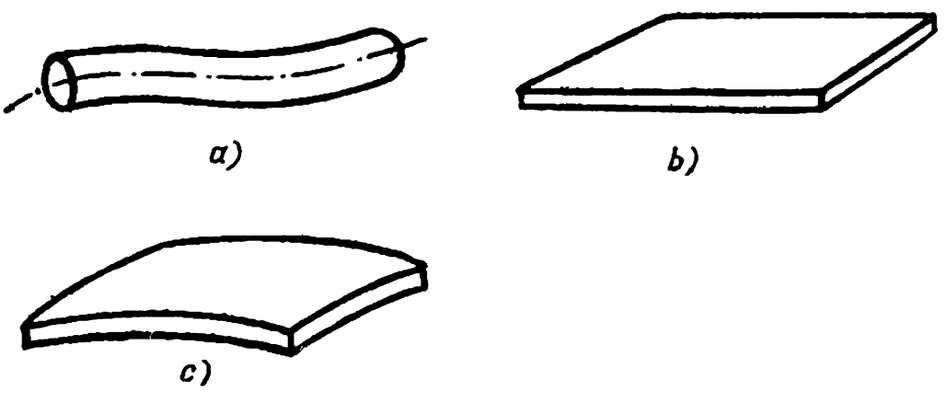


Fig. 1.1

2. A *plate* is a body bounded by two flat surfaces the distance between which is small as compared with the other dimensions (Fig. 1.1*b*).

3. A *shell* is a body bounded by two curvilinear surfaces the distance between which is small as compared with the other dimensions (Fig. 1.1*c*).

4. A *block* is a body all three dimensions of which are of the same order.

This course is concerned primarily with bodies having the form of rods of constant cross section and the simplest systems composed of such rods. It deals with rods having a considerable degree of stiffness, i. e., rods which do not become noticeably deformed under load.

In very slender bars (Fig. 1.2) such large deformations occur that they cannot be disregarded even in determining the reactions of supports. The determination of a new distance l_2, however, which differs considerably from the original one l_1, is a rather complicated problem.

Methods of analysis of slender bars, plates, shells and blocks are treated in the course on the theory of elasticity

free of the simplifying hypotheses which are introduced in the course on strength of materials. The methods of the theory of elasticity provide exact solutions of problems treated in the course on strength of materials as well as solutions of more complicated problems where it is not possible to state aplicable simplifying hypotheses.

Methods of designing bar systems are studied in the course on the theory of structures (also known as structural mechanics). The development of the science of strength of materials as well as of the above related sciences is closely bound up with industrial progress.

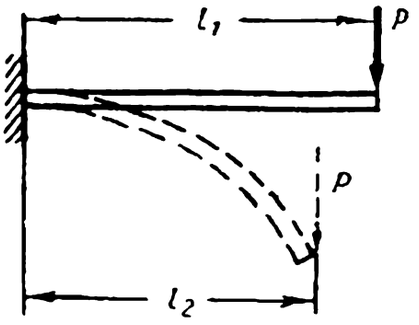


Fig. 1.2

The science of strength of materials originated in the 17th century as a result of Galileo's work. An important contribution to the development of strength of materials and theory of elasticity was made by a number of outstanding scientists—Robert Hooke, Jacob Bernoulli, Barré de Saint-Venant, Augustin Cauchy, Gabriel Lamé and others—who formulated basic hypotheses and gave several design equations.

Of particular note is the remarkable research (18th cent.) done by the eminent scientist Leonhard Euler, a member of the St. Petersburg Academy of Sciences. Wide use is still made of his work on stability analysis of compressed members.

In the 19th century the work of the Russian scientists D. I. Zhuravsky, K. S. Golovin and others acquired world importance. D. I. Zhuravsky's formula for determining shearing stresses in bending is still in use today.

Important and interesting investigations on stability analysis of compressed members which have not lost their significance nowadays were carried out by F. S. Yasinsky at the end of the 19th century.

At the beginning of the 20th century Russian scientists began to make a still greater contribution to the theory of strength of materials. Valuable works were written by Professor I. G. Bubnov, Academician A. N. Krylov, Academician B. G. Galerkin and others further developing and refining methods in this field. Bubnov-Galerkin's methods

for solving complex problems in strength of materials are known throughout the world.

A very great contribution to the development of the theory of strength of materials was made by Professor S. P. Timoshenko, the author of several first-class textbooks and numerous publications dealing with structural design based on strength, stability and vibration.

An extensive network of research institutes dealing with structural design has been created in the U.S.S.R. In conjunction with our establishments of higher learning these institutes have solved many problems vital to technical progress, and developed new effective methods of designing parts of complicated shape under different loading conditions. It is in point to mention here the works of Academician N. N. Davidenkov on theories of strength, those of Academician S. V. Serensen concerned with the study of strength of parts under repeated loading, and those of Academician A. N. Dinnik on stability, among others.

Of particular note are the outstanding works of Professor V. Z. Vlasov on the analysis of thin-walled bars and shells, which are widely used in engineering today.

Important investigations have been carried out by Y. N. Rabotnov, A. A. Ilyushin, E. I. Grigolyuk, N. I. Bezukhov, V. V. Bolotin, S. D. Ponomarev, A. R. Rzhanitsyn, A. F. Smirnov, A. S. Grigoryev, I. A. Simvulidi, S. N. Sokolov and other Soviet scientists.

2. Assumptions in Strength of Materials

In view of the complexity of structural analysis certain simplifying assumptions are made in strength of materials concerning properties of materials, loads and the nature of the interaction of a part and loads.

Experimental verification of design relations obtained on the basis of the assumptions given below shows that the resultant error is so insignificant that for practical purposes it can be neglected.

Assumption 1. *The material of the body has a solid (continuous) structure.* This assumption is fully justified from the practical point of view as most structural materials have such fine-grained structure that they can be considered solid, continuous, without giving rise to appreciable

error. Calculations based on the assumption of solid structure give satisfactory results in practice even for such materials as timber, concrete and stone.

This is due to the fact that the dimensions of real parts are many times greater than interatomic distances.

The above assumption makes it possible to employ a method of analysing infinitesimal volumes for which the mathematical apparatus of continuous functions may be used and to apply the results obtained to real specimens.

Assumption 2. *The material of a part is homogeneous, i.e., it has identical properties at all points.* Metals possess a high degree of homogeneity, i. e., they have practically the same properties throughout a part. Timber, concrete, stone and reinforced plastics are less homogeneous. For example, concrete contains an aggregate—small stones, gravel and brick—the properties of which are different from those of cement. In timber there are knots whose properties also differ greatly from the properties of the remainder. In plastics the properties of a resin differ from those of a filler.

Nevertheless, as experience shows, calculations based on the assumption of homogeneity of the material of a part give satisfactory results for main structural materials.

Assumption 3. *The material of a part is isotropic, i.e., it has identical properties in all directions.* Investigations show that the crystals of which many materials consist have quite different properties in different directions. For example, copper crystals are more than three times as strong in one direction as in another.

However, in fine-grained materials the properties in different directions become uniform, they "level out" due to the disorderly arrangement of a great number of crystals, and these materials can be considered virtually isotropic.

For such materials as timber, reinforced concrete and plastics the above assumption is only approximate.

Such materials (whose properties are different in different directions) are called *anisotropic.*

In the solution of some problems, such as those concerned with the use of plastics, it is necessary to take into account the anisotropy of a material by the application of the methods of the theory of elasticity.

Assumption 4. *There are no internal (initial) forces in a body prior to loading.* The forces of interaction between the particles of a material, the distances between which vary, resist changes in the shape and dimensions of the body under load. Henceforth, speaking of internal forces (or forces of elasticity) we shall have in view just these forces, without taking into account the molecular forces existing in an unloaded body as well.

This assumption is not, strictly speaking, true of a single material. In steel parts there exist internal forces induced non-uniform cooling, and in timber by non-uniform drying; in concrete they arise during setting.

The magnitude of these forces is usually unknown to a designer. When there is reason to suppose that these forces are considerable an attempt is made to determine them (experimentally).

Assumption 5, or the principle of superposition. *The effect of a system of forces acting on a body is equal to the sum of the effects of these same forces applied to the body in succession and in any order.* The word "effect" implies deformations, internal forces produced in the body and displacements of individual points, depending on the particular case.

It should be borne in mind that the action of the separate forces of a system should be considered in conjunction with the corresponding reactions of constraints.

The principle of superposition, extensively used in theoretical mechanics for absolutely rigid bodies, can be applied to deformable bodies only under the following two conditions:

1. Displacements of the points of application of forces are small compared to the dimensions of a body.
2. Displacements resulting from the deformation of a body depend linearly on the acting forces.

In ordinary structures both these conditions are fulfilled, and therefore the principle of superposition is widely used in structural design.

Assumption 6, or Saint-Venant's principle. *At points of a body which are sufficiently distant from the places of application of loads the magnitude of internal forces depends to a very small extent on the particular manner in which these loads are applied.*

This principle permits the replacement of one system of forces by another, statically equivalent system which may simplify the analysis.

For example, in the calculation of a rail (Fig. 1.3) as a bar resting on many supports (sleepers) it is possible to replace the actual load due to the wheel distributed over the contact area according to a certain law (which is rather difficult to find) by a concentrated force.

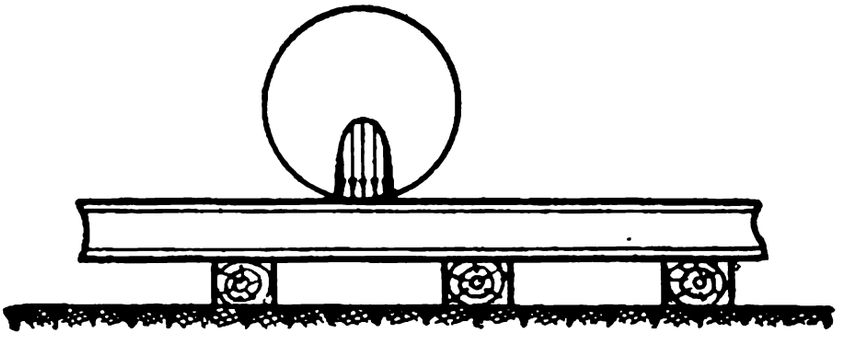


Fig. 1.3

Theoretical and experimental investigations show that in determining internal forces at points distant more than 1.5-2 times the greatest dimension of the contact area from the region of loading one load can in fact be replaced by another, statically equivalent one.

Some other assumptions and hypotheses will be stated subsequently, in appropriate parts of the course.

3. Systems of Mechanical Units

The study of strength of materials involves such quantities as force, mass, work, power, etc.

Most commonly used in engineering practice lately has been the system of units, abbreviated as mkgfs, in which the fundamental units are: the unit of length—the metre, the unit of force—the kilogram force (kgf), the unit of time—the second. This system has, however, a number of defects. First, the fundamental unit kilogram force is a variable quantity whose value depends on the geographical latitude of the location, the altitude and ranges within about 0.2 per cent. Second, the mkgfs system has been found to be absolutely unacceptable in the field of electrical and magnetic units; consequently it cannot be universal.

The Eleventh General Conference on Measures and Weights (October 1960) took a decision to establish a universal practical system of units based on six fundamental units: the unit of length—the metre, the unit of mass—the kilogram, the unit of time—the second, the unit of temperature—the degree on the Kelvin scale *, the unit of current—the ampere, the unit of light power—the candle-power. This system of units was called the international system of units, the international symbol being Si.

In the branch of mechanics and molecular physics the symbol mks is also used besides the symbol Si.

With reference to problems of strength of materials the most important distinction of the Si system from the mkgfs system is that in the Si system the mass unit kg is regarded as fundamental and the force unit as derived.

In the Si (mks) system the unit of force is taken to be the force which gives a mass of 1 kg an acceleration of 1 m/sec². This unit is called the newton and designated as N.

The units of other quantities (moments, stresses, power, etc.) are changed accordingly.

In order to pass from the mkgfs system to the Si system and vice versa it is necessary to use the following relations:

force

$$1 \text{ kgf} = 9.80665 \text{ N} \approx 10 \text{ N}$$
$$1 \text{ N} = 0.10197 \text{ kgf} \approx 0.1 \text{ kgf}$$

mass

$$1 \text{ kgf sec}^2/\text{m} \approx 9.81 \text{ kg} \approx 10 \text{ kg}$$
$$1 \text{ kg} \approx 0.1 \text{ kgf sec}^2/\text{m}$$

pressure

$$1 \text{ kgf/cm}^2 \approx 10 \text{ N/cm}^2 = 10^5 \text{ N/m}^2$$
$$1 \text{ N/cm}^2 \approx 0.1 \text{ kgf/cm}^2.$$

A convenient unit of pressure is the out-of-system unit bar equal to 10^5 N/m² **. With the same unit used for the measurement of stresses the passage from the mkgfs system to the Si system may easily be effected as 1 kgf/cm² = 1 bar.

---

* Zero on the Kelvin scale corresponds to —273° C (precisely —273.16° C).

** GOST 7664-61.

work (energy)

$$1 \text{ kgf m} \approx 10 \text{ N m} = 10 \text{ j}$$
$$1 \text{ N m} = 1 \text{ j} \approx 0.1 \text{ kgf m}.$$

The unit of work in the Si (mks) system is taken to be the work done by a force of 1 newton over a distance of 1 metre. This unit of work is called the joule and designated as j.

power

$$1 \text{ kgf m/sec} = 10 \text{ N m/sec} = 10 \text{ j/sec} = 10 \text{ w}$$
$$1 \text{ hp} = 75 \text{ kgf m/sec} = 736 \text{ w} = 0.736 \text{ kw}$$
$$1 \text{ kw} = 1000 \text{ j/sec} = 1000 \text{ N m/sec} = 100 \text{ kgf m/sec}.$$

The information on the dimension of other mechanical quantities in the Si system is presented in Appendix III. Appendix II gives the notation of divisions and functions of a unit which are obtained by adding to the primary unit the words: micro-, milli-, hecto-, mega-, etc.

4. External Forces (Loads)

Loads acting on structures and their elements represent forces or couples (moments) that can be considered *concentrated* or *distributed.*

Strictly speaking, there exist no concentrated forces in nature. All real forces are forces distributed over some area. For example, the pressure of a wheel on a rail is actually transmitted through a small area resulting from the deformation of the rail and the wheel (Fig. 1.3). However, in determining the internal forces induced in the rail and the wheel at a certain distance from the area of pressure transmission it is possible on the basis of the foregoing Saint-Venant's principle to replace the distributed load by the resultant concentrated force, thus simplifying the analysis.

But in the immediate vicinity of the contact area of two bodies internal forces, in this case called contact forces, cannot be determined by the methods of strength of materials. They are computed by the methods of the theory of elasticity.

Concentrated loads are measured in tons or kilograms (or in newtons in the Si system).

Distributed loads may be in the form of surface tractions (for instance, wind or water pressure against a wall) and body forces (for instance, the weight of a body).

The weight of a bar is commonly treated as a load distributed along its length (linear load) rather than over its volume taking into account small dimensions of the cross section compared with the length.

Distributed loads are measured in units of force: tons, kilograms (newtons in the Si system) related to unit of length or to unit of area or to unit of volume. Both concentrated and distributed loads can be static as well as dynamic.

Static loads are those which change their magnitude or point of application (or direction) at a very small rate, so that accelerations occurring in this case can be neglected.

Under such loading conditions vibrations of structures and their parts are negligible.

Dynamic loads are those which vary in time with great speed (for example, impact loads). The action of such loads sets up vibration. Under vibrations, due to changes in the speed of vibrating masses, inertia forces arise which are proportional (according to Newton's second law) to the vibrating masses and accelerations. The magnitude of these inertia forces may be many times the static loads.

The laws governing the change of loads with time may be very complicated in character.

In a particular case the change in a load may be of a periodically repeating character, the maximum values of the load being continually repeated after the same interval of time. Loads of this type are called loads with steady conditions or *repeated periodic loads* (Fig. 1.4). Strength analysis under such loading conditions is given in Chapter XII.

However, in many other cases the change in a load with time is not of a steady character (Fig. 1.5). Such are loads acting on parts of automobiles, tractors, machines, and also loads acting on structures (buildings, masts, etc.) due to pressure of wind, snow and so on. These are called *repeated loads of unsteady rates.* A more profound study of such loading conditions can only be carried out with the aid of statistical methods and the theory of probability, which are used for analysing random quantities.

By way of an example consider a load due to wind. It is known that the wind velocity, which determines the wind load, is constantly changing at one and the same geographical point. For example, according to observations in the Moscow Region over a long period of time wind velocity varied over a very wide range (Fig. 1.6).

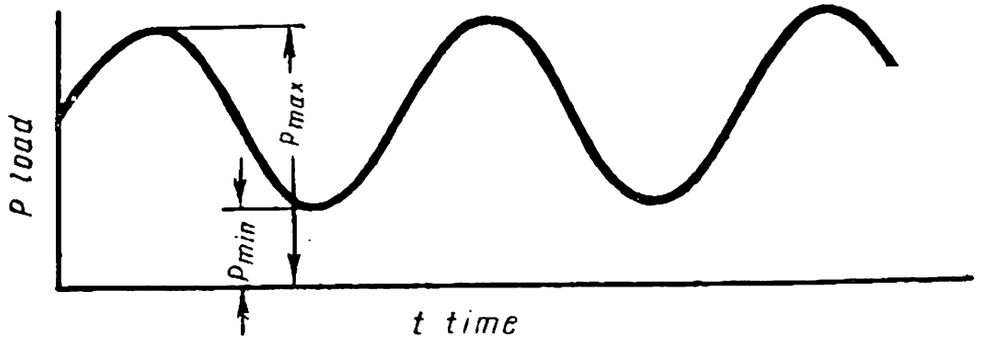


Fig. 1.4

A velocity of 3.5 m/sec was observed most frequently (33 per cent of all cases). However, there were instances when it reached 12 m/sec (2 per cent of all cases) and more. On the other hand, the velocity was sometimes less,

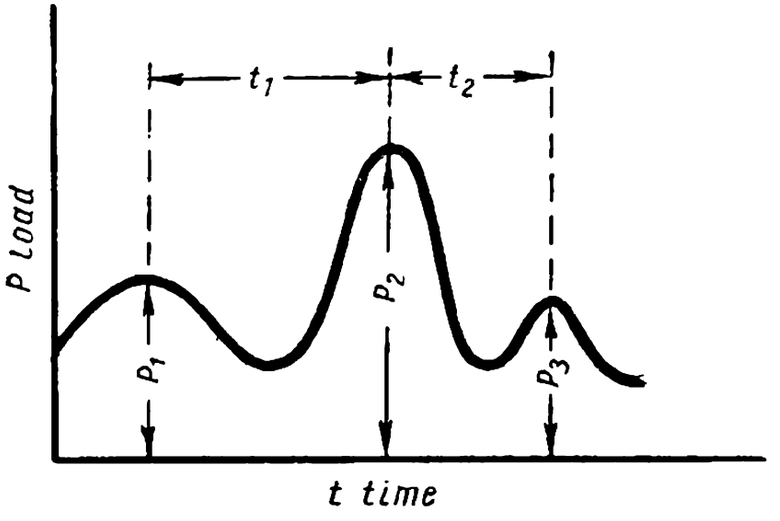


Fig. 1.5

and occasionally equal to zero (which was extremely seldom).

Curves similar to that considered above are called *distribution curves.* They provide a visual representation of the dispersal of a given quantity.

What wind velocity should be taken in design? The maximum on record immediately springs to mind. But,

first, there is no guarantee that during its lifetime the structure will not be subjected to a stronger wind than has ever been registered before. Secondly, it is obviously uneconomic to take a wind velocity that recurs once in 200 or 100 years in designing a short-life structure (for instance, a wooden building). Consequently, the value of a design load must be closely related to the lifetime of the structure and its purpose.

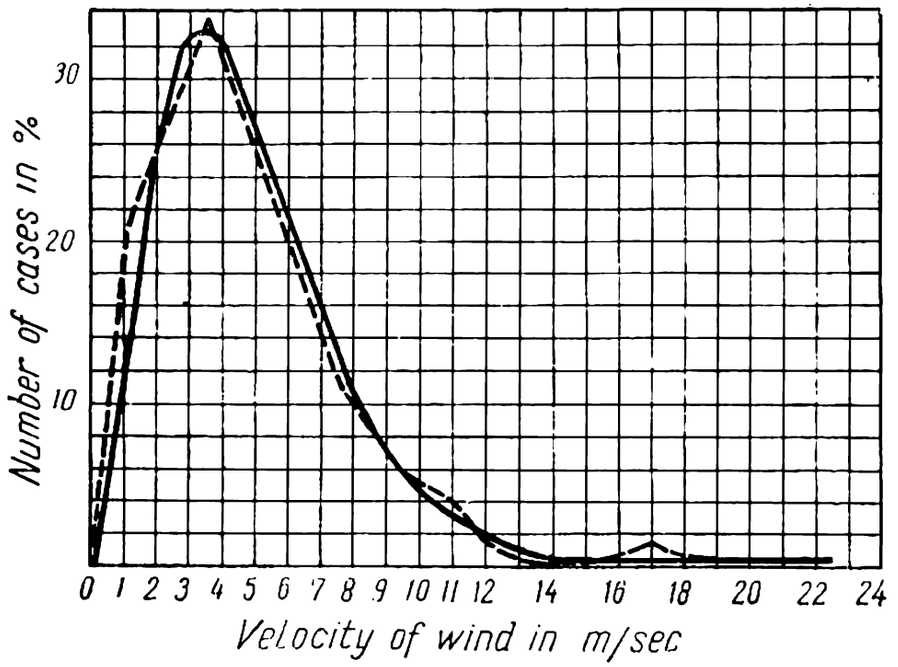


Fig. 1.6

All that is said about wind load refers equally to most other loadings. *

In structural analysis values of design loads are governed by specifications and building codes.

In mechanical engineering design loads depend on the service conditions of the particular machine: rated capacity, number of revolutions of its individual parts, its own weight, inertia forces, etc. For instance, in designing parts of a three-ton automobile a nominal payload of three tons is taken into consideration. Possible overloading of the automobile is allowed for by prescribing cross-sectional dimensions of parts with a certain factor of safety. The value of this safety factor will be discussed in more detail in Section 12.

* For further details in regard to this question the reader is referred to the textbook by V. V. Bolotin *Statistical Methods in Structural Mechanics* (Moscow, Stroiizdat, 1960).

5. Deformations and Displacements

As has been pointed out previously, all bodies subject to external forces *deform* to some extent, i.e., change their dimensions or shape or both simultaneously.

The variation in the linear dimensions of a body is called linear deformation, the variation in the angular dimensions is called angular deformation.

The increase in the dimensions of a body is called *elongation*, and the decrease in dimensions is called *contraction.*

If deformations vary over the volume of a body concern is with a *deformation at a given point* of the body in a particular direction.

If a very small rectangle *1*, *2*, *3*, *4* (Fig. 1.7*a*) is marked out on the surface of a body near the point under investi-

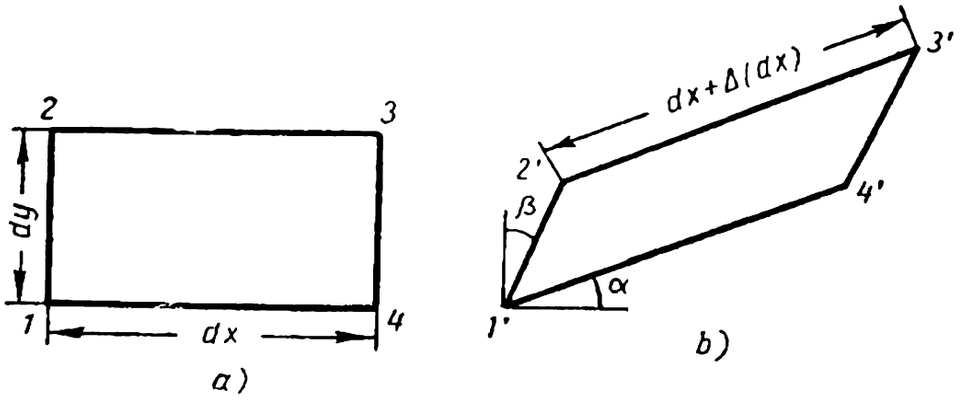


Fig. 1.7

gation, it will generally assume the form of a parallelogram *1'*, *2'*, *3'*, *4'* (Fig. 1.7*b*) due to deformation. The sides of the rectangle change in length and become inclined.

The variation in the length of the side *1 2* or *3 4* characterises the total or absolute linear deformation at the given point in the vertical direction, the variation in the length *2 3* or *1 4* characterises the total linear deformation in the horizontal direction.

The variation in the initially right angle between the sides of the rectangle being considered, $\gamma=\alpha+\beta$, characterises the angular deformation (or the shear angle) at the given point.

Absolute linear deformation is denoted by $\Delta(dz)$, $\Delta(dy)$, Δl, etc., depending on how the length of the segment concerned is denoted.

In order to eliminate the effect of the length of the sides of a rectangle it is customary to introduce the concept of unit linear deformation (strain)

$$\varepsilon = \frac{\Delta(dz)}{dz} \quad \text{or} \quad \varepsilon = \frac{\Delta(dy)}{dy} \quad \text{or} \quad \varepsilon = \frac{\Delta l}{l}.$$

Unit linear deformation is a dimensionless quantity. Linear deformation corresponding to an increase in the length of a segment is considered to be positive. Experience shows that both linear and angular deformations may disappear either completely or partially upon removal of the load (depending on the material and degree of loading).

Deformations which disappear after unloading are called elastic, and the property of bodies to recover their original shape after unloading is called elasticity.

Deformations which the body retains even after removal of the load are called permanent or plastic deformations, and the property of materials to yield permanent set is called plasticity.

If the deformations of a body at all its points and supporting conditions are known, the displacements of all the points of the body can be determined, i.e., their positions (new co-ordinates) after deformation can be indicated. For a structure to give normal service deformations of its individual elements should as a rule be elastic, and should not exceed certain permissible values. These conditions, expressed in the form of equations, are called *conditions of rigidity*. In special cases small plastic deformations can be allowed.

6. Method of Sections

Internal forces (forces of elasticity) arising inside a body under loading are assumed to be distributed continuously in accordance with the previously mentioned assumption on the continuity of the material of a body.

It will be shown later how these forces can be determined at any point of the body.

Now we shall see how to find resultant forces (including moments) to which these forces of elasticity reduce at the section,

For determining resultant internal forces (or internal force factors) *the method of sections* is used, which consists in the following.

For a body in equilibrium (Fig. 1.8) we imagine a section being made at a certain place, for instance, along *a-a*. Then one of the portions is removed (generally that to which a greater number of forces are applied). The interaction of the portions is replaced by internal forces which balance the external forces acting on the cut portion. If the external forces lie in one plane, it is necessary in general to apply three internal forces at the section to balance them: a force N acting along the axis of the bar,

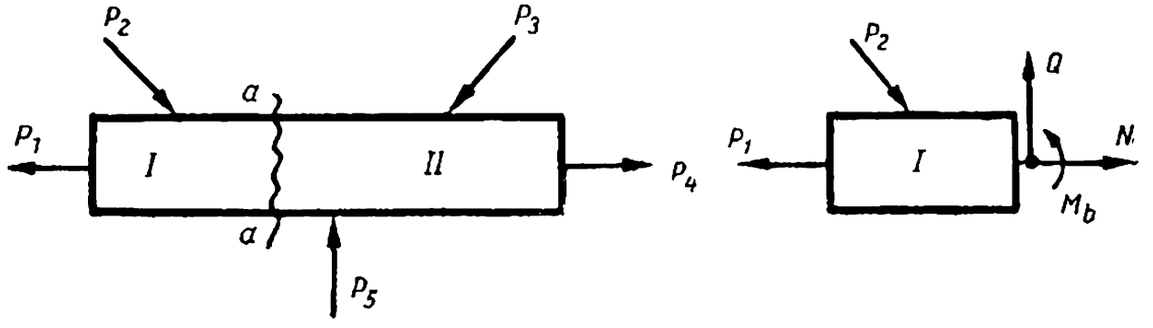


Fig. 1.8

called *the longitudinal force*; a force Q acting in the plane of the cross section and called *the transverse force*, and a moment M_b whose plane of action is perpendicular to the plane of the section. This moment arises due to the bending of the bar and is called *the bending moment.* After that equations of equilibrium for the cut portion are set up from which N, Q and M_b are determined.

In fact, by projecting the forces acting on the cut portion on the direction of the axis of the bar and setting the sum of the projections equal to zero, we find N; projecting the forces on a direction perpendicular to the axis of the bar we determine Q; equating to zero the sum of moments with respect to any point we obtain M_b.

If the external forces with the reactions of the supports included do not lie in one plane (three-dimensional problem), six internal forces may arise at a cross section in the general case, which are components of the resultant force vector and the resultant moment vector of the system of internal forces (Fig. 1.9): a longitudinal force N, a transverse force Q_y, a transverse force Q_x, and three moments;

M_x, M_y, and M_z, the first two being bending moments and the third M_z, acting in the plane of the section, being called *the twisting moment*, as it is caused by torsion of the bar. For determining these six forces it is necessary to use six equations of equilibrium: equate to zero the sums of projections of the forces (applied to the cut portion) on three co-ordinate axes, and also the sums of moments of the forces about three axes having the origin at the centroid of the section.

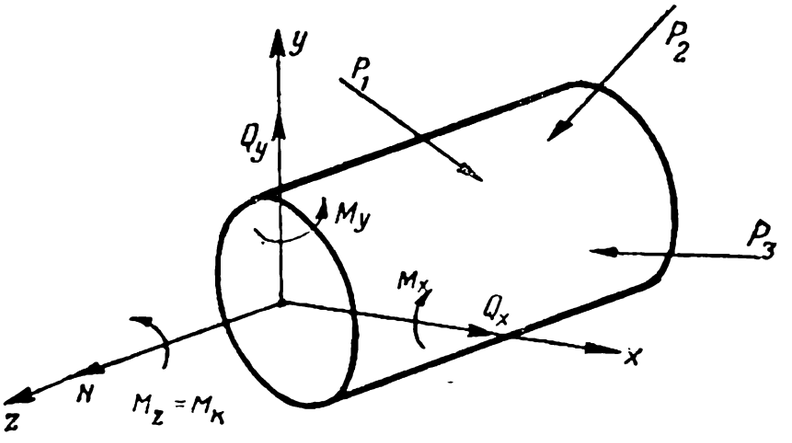


Fig. 1.9

In Fig. 1.9 and henceforth the right-handed co-ordinate system is adopted, the z-axis being, as a rule, coincident with the axis of a bar.

Thus, to find internal forces it is necessary:

(1) to cut a bar or a system of bars;

(2) to remove one portion;

(3) to apply at the section forces capable of balancing the external forces acting on the cut portion;

(4) to find the magnitudes of the forces from the equations of equilibrium derived for the cut portion.

In particular cases the following forces may arise at cross sections of the bar:

1. *A longitudinal force N only. This case of loading is called tension (if the force N acts away from the section) or compression (if the longitudinal force acts toward the section).*

2. *A transverse force Q_x or Q_y only. This is a case of shear.*

3. *A twisting moment M_t only. This is a case of torsion.*

4. *A bending moment M_x or M_y only. This is a case of bending.*

5. *Several forces, for instance bending and twisting moments. These are cases of complex deformations (or complex resistance) which will be discussed at the end of the book.*

7. Stresses

It has been noted but it may be well to emphasise once again that at a cross section of a bar there act continuously distributed forces whose intensity may be different at different points of the section and in different directions rather than concentrated internal forces N, Q, M_t, etc.

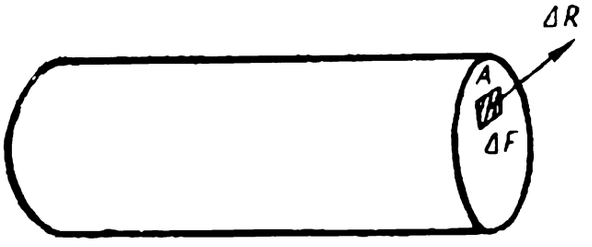


Fig. 1.10

And how can the intensity of internal forces be measured at a given point of a given section, say, at a point A of a section I? (Fig. 1.10).

Isolate a small area ΔF enclosing the point A. Let ΔR be the resultant of the internal forces acting upon this area.

Then the average magnitude of the internal forces per unit area of ΔF is

$$p_{av} = \frac{\Delta R}{\Delta F}. \tag{1.1}$$

The quantity p_{av} is the *average stress.* It characterises the average intensity of the internal forces. By reducing the dimensions of the area we obtain in the limit

$$p = \lim_{\Delta F \to 0} \frac{\Delta R}{\Delta F}. \tag{1.2}$$

The quantity p is the true stress, or simply *stress*, at the given point of the given section.

Briefly, it may be said that *the stress is the internal force per unit area at a given point of a given section.*

As can be observed from formulas (1.1) and (1.2), the dimension of stress is $\frac{[\text{load}]}{[\text{area}]}$.

In the mkgfs system the dimension of stress is kgf/m^2. It is common practice to use the out-of-system units kgf/cm^2 or kgf/mm^2.

In the Si system the dimension of stress is $N/m^2 = \frac{m \cdot kg \cdot sec^{-2}}{m^2} = m^{-1} \cdot kg \cdot sec^{-2}$.

Since real values of stresses are expressed by very large numbers, it is necessary to employ multiple values of units, for example $\frac{\text{meganewton}}{m^2}$, or concisely $\frac{MN}{m^2} = 10^6 N/m^2$.

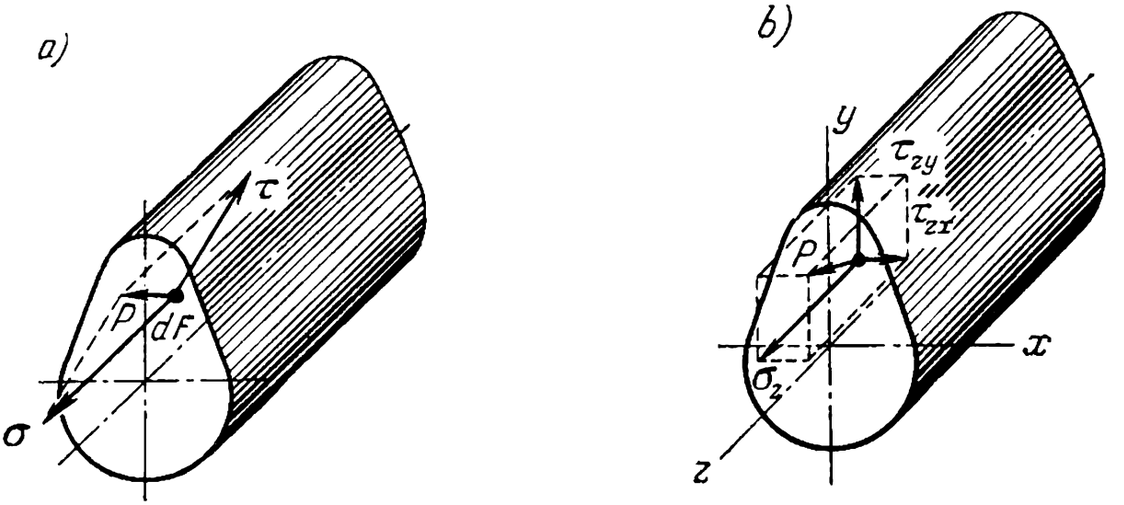


Fig. 1.11

The total stress p can be resolved into two components (Fig. 1.11*a*):

1. A component normal to the plane of the section. *This component is usually denoted by* σ *and called the normal stress.*

2. A component which lies in the plane of the section. *This component is usually denoted by* τ *and called the shearing stress.* The shearing stress can have any direction in the plane of the section depending on the acting forces. For convenience τ is usually represented in the form of two components in the direction of the co-ordinate axes (Fig. 1.11*b*).

The common symbols used for stresses are shown in Fig. 1.11*b*. The normal stress has a subscript indicating the co-ordinate axis to which this stress is parallel. Tensile normal stress is considered as positive, and compressive normal stress as negative. The symbols of shearing stresses have two subscripts: the former indicates the axis to which the normal to the area of action of the given

stress is parallel and the latter indicates the axis to which the stress itself is parallel.

The decomposition of the total stress into normal and shearing stresses has a definite physical meaning. The normal stress occurs when the particles of the material which are in contact over a given area tend to move away from each other or, on the contrary, toward each other. The shearing stresses are associated with the slip of particles of the material in the plane of the section considered.

Imagine an infinitely small cube enclosing any point of a body being isolated; generally speaking, its faces will be acted upon by the stresses represented in Fig. 1.12.

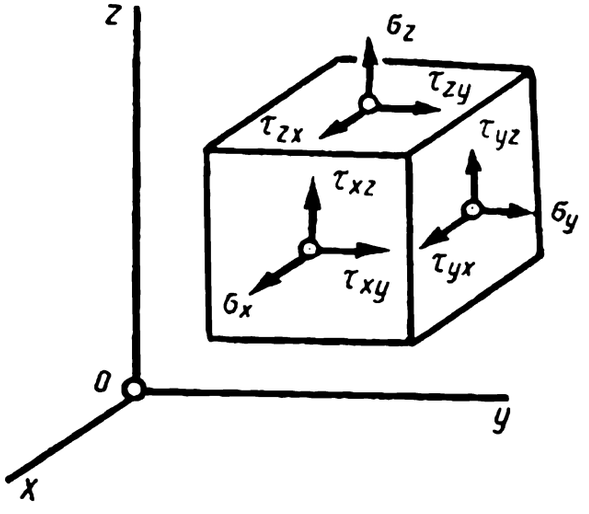


Fig. 1.12

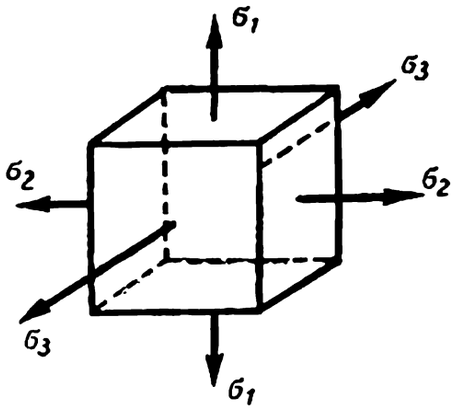


Fig. 1.13

A set of stresses on all the elementary areas which can be passed through any point of a body is called the state of stress at the given point.

If the normal stresses alone act upon the faces of the cube, they are called principal and the areas they act upon are called principal areas.

In the theory of elasticity it is proved that there exist three principal areas (at right angles to each other) at each point of a stressed body.

The principal stresses are usually denoted by σ_1, σ_2 and σ_3, the greatest principal stress (with regard to sign) being denoted by σ_1 and the smallest by σ_3.

Various kinds of stress state are usually classified according to the number of acting principal stresses.

If all three principal stresses are different from zero, the state of stress is called *triaxial* or *three-dimensional*

(Fig. 1.13). If one of the principal stresses is zero, the state of stress is called *biaxial* or *plane*. If two principal stresses are zero, the state of stress is called *uniaxial* or *linear*.

If the state of stress is known at any point of a part, the strength of this part can be estimated.

In many cases an estimate of the strength of structural elements having the form of a bar is made on the basis of the value of the maximum stresses arising at their cross sections. In the simplest case, when the *linear* state of stress occurs at the dangerous point of the cross section, *the condition of strength* is written as

$$\sigma \leqslant [\sigma] \tag{1.3}$$

where σ is the actual stress arising at the point considered and $[\sigma]$ is the allowable stress whose magnitude depends on the properties of the material (determined experimentally) and the service conditions of the structural element being designed. Data concerning the determination of the values of allowable stresses are given in Section 12.

There are cases when an estimate of strength is made on the basis of shearing stresses, the condition of strength being written in a similar way to (1.3), as

$$\tau \leqslant [\tau]. \tag{1.4}$$

It will be shown later that in some cases it is more correct to set up the condition of strength for a part on the basis of loads rather than stresses as the attainment of the ultimate value of stress at the most dangerous point does not always mean failure of the part.

Example 1.1. Determine the forces in the bars AB and BC of the system shown in Fig. 1.14.

Solution. To determine the forces in the bars AB and BC apply the method of sections. Draw a section a-a across the bars, remove the left-hand portion and consider the equilibrium of the right-hand portion.

First assume the forces in both bars to be tensile (tensile forces are directed away from the joint in the drawing) and denote them by N_1 and N_2. Derive the equilibrium equations for the cut portion of the system

$$\Sigma Y = 0; \quad -P - N_2 \sin \alpha = 0. \quad \text{Hence} \quad N_2 = -\frac{P}{\sin \alpha}.$$

The negative sign indicates that the force N_2 is not tensile, as has been assumed, but compressive. Derive the second equation of equilibrium

$$\Sigma X = 0; \quad -N_1 - N_2 \cos\alpha = 0.$$

Substituting the value $N_2 = -\frac{P}{\sin\alpha}$ we obtain $N_1 = P\cot\alpha$.

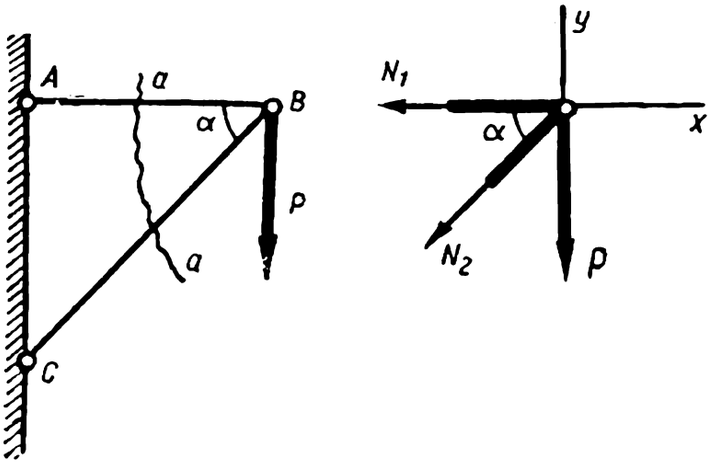


Fig. 1.14

Example 1.2. Determine the forces in the bars AB and CD of the system shown in Fig. 1.15.

Solution. Consider the equilibrium of the portion of the system below the section. Equating to zero the sum of

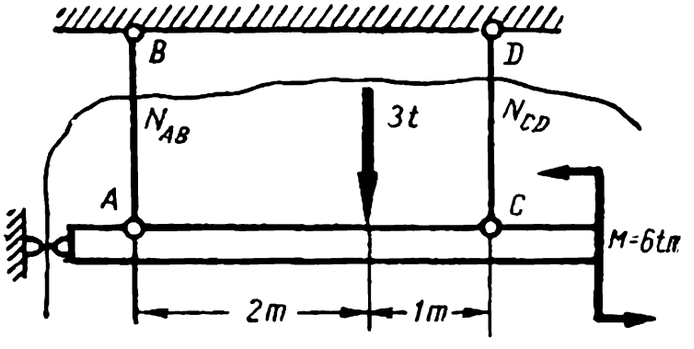


Fig. 1.15

projections on a horizontal axis, we see that the force in the horizontal supporting bar (at point A) is zero.

Equating to zero the sum of moments of all forces acting on the cut portion with respect to A, we obtain

$$\Sigma M_A = 0; \quad -3 \cdot 2 + N_{CD} \cdot 3 + 6 = 0. \quad \text{Hence} \quad N_{CD} = 0.$$

Equating to zero the sum of projections of the same forces on a vertical line, we obtain

$$\Sigma Y = 0; \quad N_{AB} - 3 = 0. \quad \text{Hence} \quad N_{AB} = 3\text{t}.$$

The positive sign indicates that the force N_{AB} acts as shown in the drawing, i.e., it is tensile.

The use of the international system of units (Si system) would entail the following alterations. The load of 3 tons would be $3 \cdot 10^4$ N, instead of the moment $M = 6$ tm we would have $M = 6 \cdot 10^4$ Nm. The force N_{AB} would be given (with the same equilibrium equations) by $N_{AB} = 3 \cdot 10^4$ N.

CHAPTER II

TENSION AND COMPRESSION

8. Determination of Internal Forces

Consider the case of axial (central) tension or compression when external forces act along the axis of a bar (Fig. 2.1). For determining internal forces (longitudinal forces) the method of sections will be applied.

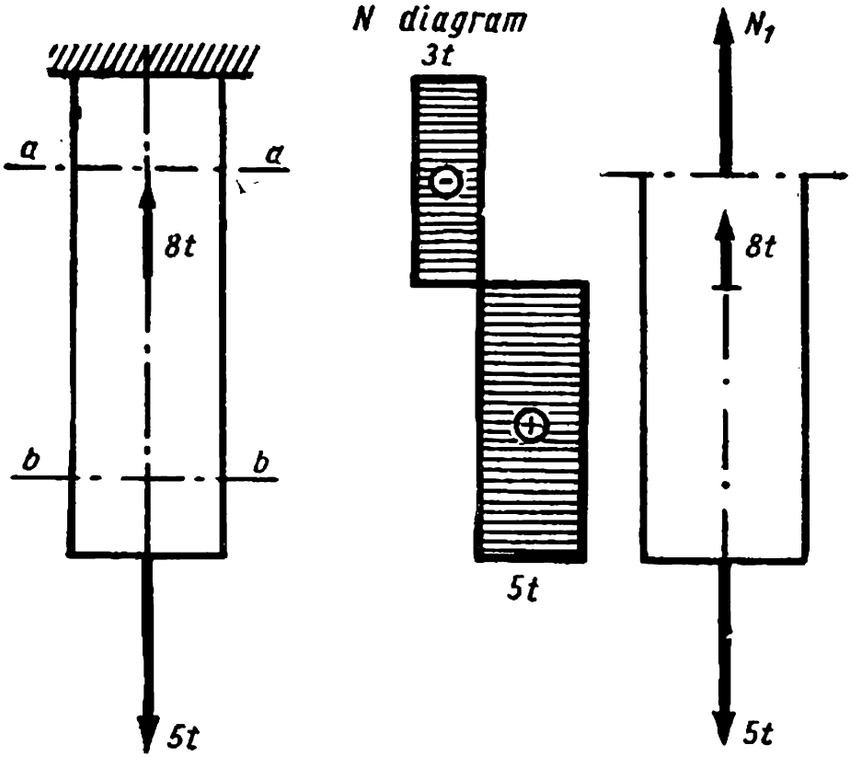


Fig. 2.1

Draw any section, for instance *a-a*, and consider the equilibrium of the lower cut portion. Replace the action of the upper removed portion on the lower portion by a longitudinal force and direct it away from the section, i.e., assume that the force is tensile. Derive the equation of equilibrium. Projecting all forces acting on the lower portion on the direction parallel to the axis of the bar and

setting the sum of the projections equal to zero, we obtain $N_1+8-5=0$, hence $N_1=-3$ t.

The minus sign indicates that the direction of the force N_1 should be reversed, i.e., the longitudinal force will be compressive in this case and not tensile as we have assumed. In a similar way we shall find the longitudinal force at the section b-b : $N_2=5$ t (tension). A longitudinal force corresponding to tension will be considered positive.

A visual representation of the law governing the change in the longitudinal forces along the bar length is given by a graph *(longitudinal force diagram)* in which the axis of abscissas is drawn parallel to the axis of the bar and the ordinate axis is perpendicular to it. The values of the longitudinal forces (with regard to sign) at cross sections of the bar are plotted to a selected scale along the ordinate axis. For the case considered the N diagram is shown in Fig. 2.1.

9. Determination of Stresses

If a network of lines parallel and perpendicular to the axis of a prismatical bar is marked out on the surface of the bar (Fig. 2.2*a*) and a tensile force is applied to it, one can observe that the lines of the network will remain perpendicular to one another after deformation except for a small region of the bar near the point of application of the force which is eliminated from consideration for the present *, but the distances between them will be changed (Fig. 2.2*b*). All the horizontal lines, for example *cd*, will move downwards, remaining horizontal and straight. It may be assumed that the same pattern exists inside the bar as well, i.e., *cross sections of the bar plane and normal to its axis before deformation will remain plane and normal to the axis after deformation.* This important hypothesis is called *the hypothesis of plane sections* or *Bernoulli's hypothesis.* Formulas obtained on the basis of this hypothesis are confirmed by experimental results.

Such a pattern of deformations gives grounds for assuming that only normal stresses uniformly distributed over a section act on cross sections of the bar.

---

* This is not reflected in the drawing (Fig. 2.2).

From the equilibrium condition of the portion of the bar represented in Fig. 2.2*c* we obtain $\Sigma Z=0$; $-\sigma F+P=0$ or

$$\sigma=\frac{P}{F}. \tag{2.1}$$

In the general case when several forces act on the cut portion, the algebraic sum of the projections of these

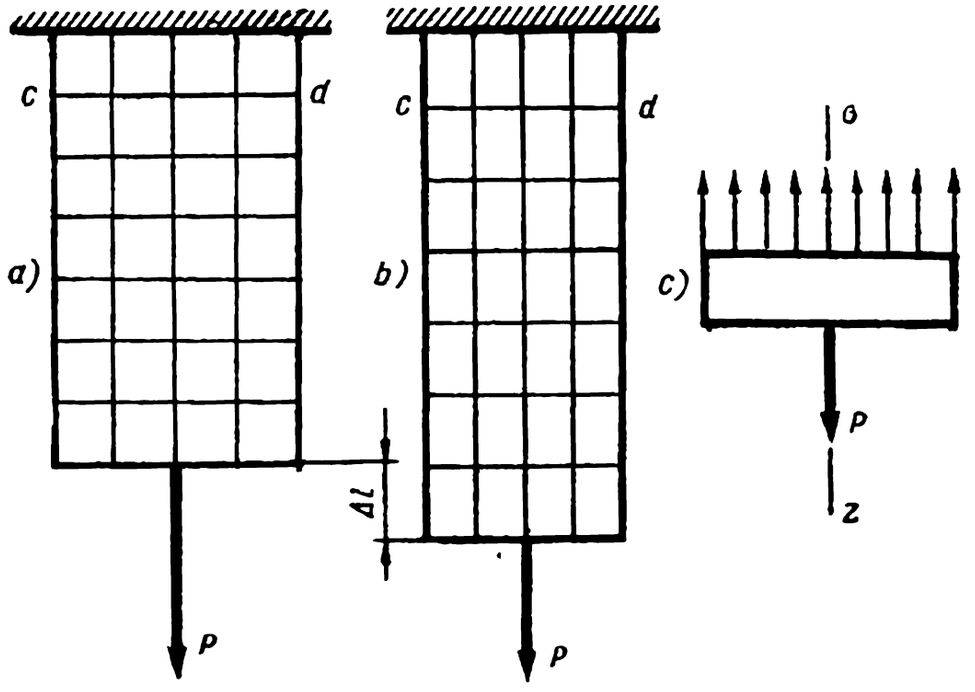


Fig. 2.2

forces on the axis of the bar, numerically equal to the longitudinal force N, enters in the numerator of formula (2.1), i.e.,

$$\sigma=\frac{N}{F}. \tag{2.2}$$

These formulas are also valid for compression, the only difference being that compressive stresses are considered negative. *

In addition to strength, stability also has to be considered in the design of compressed bars (see Chapter X).

10. Determination of Deformations and Displacements

Experiments show that under tension the length of a bar increases and lateral dimensions decrease, while the reverse happens under compression (see Fig. 2.2*b*).

* Formula (2.2) and the most important of those which follow are marked by thick numbers.

For many materials experiments give the following relation between longitudinal strain ε and stress σ under loading within certain limits:

$$\varepsilon = \frac{\sigma}{E}. \tag{2.3}$$

This relation is called *Hooke's law* and is formulated as follows: *linear strains are proportional to normal stresses.*

In formula (2.3) E is a factor dependent on the material and is called the *modulus of longitudinal elasticity* or the modulus of elasticity of the first kind. It characterises the *stiffness of a material*, i.e., its ability to resist deformation.

Since ε is a dimensionless quantity, it is evident from formula (2.3) that the dimension of E is that of σ, i.e., kg/cm^2.

Table 2.1 gives the average values of E for a number of materials.

Table 2.1

Values of the Moduli of Longitudinal Elasticity

Material	E in kg/cm^2
Steel	2×10^6 to 2.2×10^6
Copper	1×10^6
Timber	1×10^5
Aluminium	0.675×10^6
Cast iron	0.75×10^6 to 1.6×10^6
Fibre glass reinforced plastics	0.18×10^6 to 0.40×10^6

For other materials the value of E can be found in handbooks.

Bearing in mind that $\varepsilon = \frac{\Delta l}{l}$ and $\sigma = \frac{N}{F}$ a formula for determining the absolute elongation (contraction) can be obtained from expression (2.3)

$$\Delta l = \frac{Nl}{EF} \tag{2.4}$$

There exists an experimentally established relationship between longitudinal strain ε and lateral strain ε':

$$\varepsilon' = -\mu\varepsilon \tag{2.5}$$

Here μ is *the factor of lateral strain* (Poisson's ratio) characterising the ability of the material to deform laterally.

In using formula (2.5) elongation is considered positive, contraction negative. The value of μ for all materials varies over a range $0 \leqslant \mu \leqslant 0.5$.

The value of Poisson's ratio μ for most materials varies from 0.25 to 0.35 (Table 2.2).

Table 2.2

Material	μ	Material	μ
Steel	0.25-0.33	Lead	0.45
Copper	0.31-0.34	Brass	0.32-0.42
Bronze	0.32-0.35	Aluminium	0.32-0.36
Cast iron	0.23-0.27	Zinc	0.21
Glass	0.25	Stone	0.16-0.34
Concrete	0.08-0.18	Rubber	0.47
Cork	0.00	Plywood	0.07
Celluloid	0.39		

For steel the value of μ can be taken as $\mu \approx 0.3$ in the case of elastic deformations. If ε' is known the total lateral deformation Δb can be calculated from the formula

$$\varepsilon' = \frac{\Delta b}{b}, \tag{2.6}$$

where b is the original lateral dimension of the bar.

In bars of variable cross section (Fig. 2.3) stresses can be considered as uniformly distributed over cross sections (if the angle of the cone $\alpha \leqslant 12°$) and can be determined by the use of the same formula (2.2) as for a bar of constant cross section.

For determining deformations of a bar of variable cross section whose cross sections are acted upon by a longitudinal force N, first find the elongation $\Delta(dz)$ of an element

of length dz which is the differential of the total elongation Δl. According to Hooke's law we have

$$\Delta(dz) = d(\Delta l) = \frac{N\,dz}{EF}. \tag{2.7}$$

The total elongation of the bar will be obtained by integrating expression (2.7) from $z=0$ to $z=l$

$$\Delta l = \int_0^l \frac{N\,dz}{EF}. \tag{2.8}$$

If N and F are constant, then

$$\Delta l = \frac{N}{E}\int_0^l \frac{dz}{F}. \tag{2.9}$$

In order to make use of this formula, it is necessary to know the law governing the change of F as a function of z.

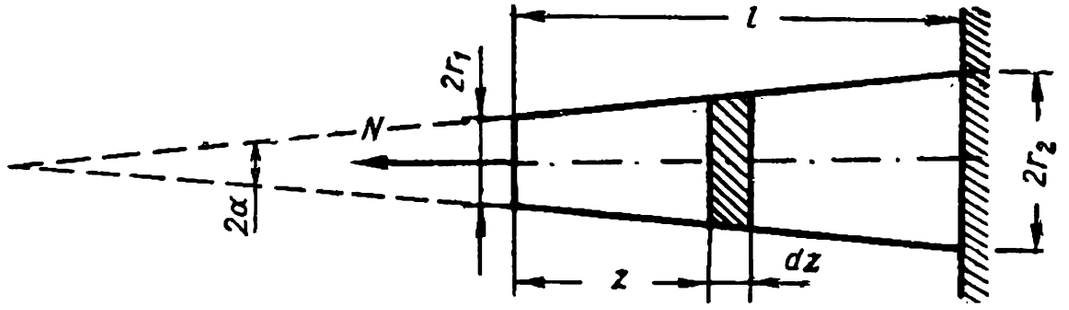


Fig. 2.3

For stepwise bars (Fig. 2.4) integration is replaced by summation and the total change in the length of the bar is found as the algebraic sum of the deformations of its separate parts within the limits of which E, N and F are constant:

$$\Delta l = \sum_{i=1}^{i=n} \Delta l_i = \sum \frac{N_i l_i}{E_i F_i}. \tag{2.10}$$

For example, for the bar shown in Fig. 2.4 we shall have

$$\Delta l = \Delta l_1 + \Delta l_2 = \frac{N_1 l_1}{E_1 F_1} + \frac{N_2 l_2}{E_2 F_2},$$

where $N_1 = N_2 = P$.

Now determine the elongation of a bar of constant cross section under its own weight which represents a load uni-

formly distributed along the bar (Fig. 2.5). Let γ denote the weight per unit volume of the material of the bar. Consider the deformation of an element dz isolated at a distance z from the lower end. It is stretched by a force γFz

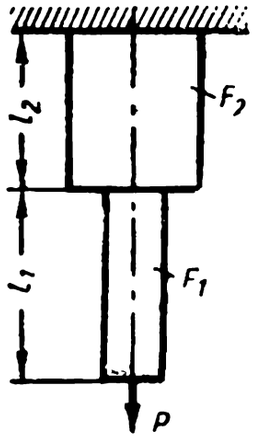


Fig. 2.4

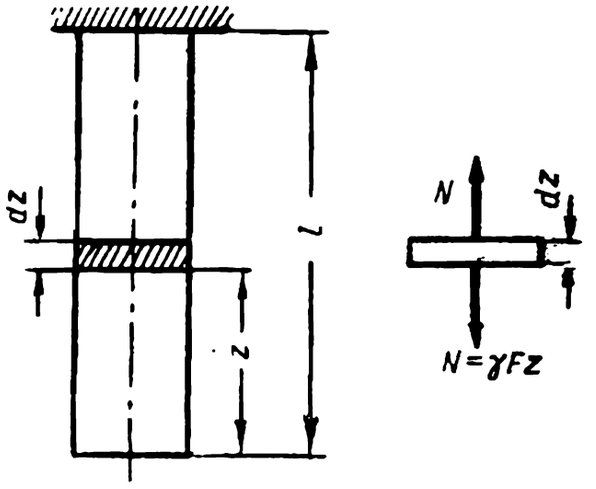


Fig. 2.5

equal to the weight of the lower portion of the bar below the section z. The elongation of the bar is

$$\Delta(dz) = d(\Delta l) = \frac{\gamma Fz\, dz}{EF} = \frac{\gamma z\, dz}{E}. \qquad (2.11)$$

Integrating this expression from $z=0$ to $z=l$, we obtain the elongation of the bar:

$$\Delta l = \int_0^l \frac{\gamma}{E} z\, dz = \frac{\gamma l^2}{2E}. \qquad (2.12)$$

This expression can be presented in another form, bearing in mind that the weight of the bar is $G=\gamma Fl$, i.e.,

$$\gamma l = \frac{G}{F}.$$

Then from (2.12) we obtain

$$\Delta l = \frac{Gl}{2EF}. \qquad (2.13)$$

Therefore, *the elongation of a bar of constant cross section produced by its own weight is half as much as is produced by a load equal to the weight of the bar and applied at its end.*

There is no great difficulty about obtaining a formula necessary for determining deformations of a bar of variable

cross section under its own weight. Its derivation is left to the student.

Example 2.1. For the steel bar shown in Fig. 2.6 determine the longitudinal force N and the stresses σ at all cross sections. Determine also the vertical displacements δ

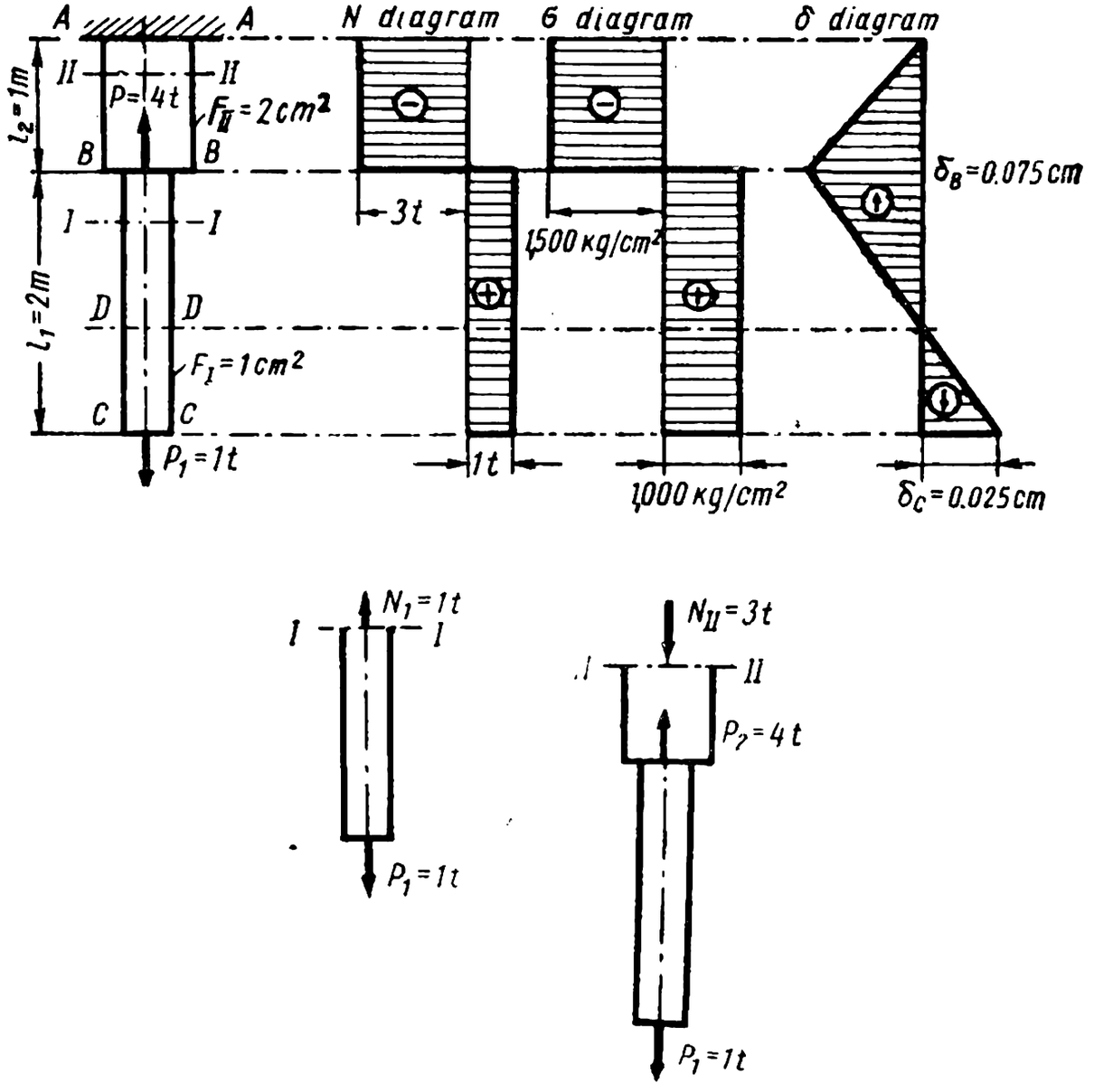


Fig. 2.6

of all cross sections of the bar. Represent the results graphically by plotting the N, σ and δ diagrams.

Solution. To determine N imagine the bar cut across sections *I-I* and *II-II*. From the equilibrium condition of the portion of the bar below the section *I-I* we have: $N_I = P_1 = 1$t (tension). From the equilibrium condition of the portion of the bar below the section *II-II* we have:

$-N_{II}+P_2-P_1=0$ or $-N_{II}+4-1=0$ or $N_{II}=3$ t (compression).

Next we construct the longitudinal force diagram choosing a certain scale, the tensile longitudinal force N being considered positive, and compressive negative. Stresses are: at the sections of the lower portion of the bar $\sigma_I=\frac{1{,}000}{1}=1{,}000$ kg/cm² (tension), at the sections of the upper portion of the bar $\sigma_{II}=-\frac{N_{II}}{F_{II}}=-1{,}500$ gk/cm² (compression).

To a certain scale we construct the stress diagram.

To plot the δ diagram we determine the displacements of the characteristic sections *B-B* and *C-C* (the displacement of the section *A-A* is equal to zero). The section *B-B* will move upward since the upper portion of the bar is in compression:

$$\delta_B=-\frac{3\,000\times 100}{2\times 10^6\times 2}=-0.075 \text{ cm (upward).}$$

The displacement of a section downward is considered positive, and upward negative.

The displacement of the section *C-C* is the algebraic sum of the displacement of the section *B-B* (δ_B) and the elongation of the bar l_1:

$$\delta_C=\delta_B+\Delta l_1=-0.075+\frac{N_1 l_1}{EF_1}=$$

$$=-0.075+\frac{1{,}000\times 200}{2\times 10^6\times 1}=0.025 \text{ cm (downward).}$$

To a certain scale we plot the values of δ_C and δ_B on the diagram, join the points obtained by straight lines, as displacements depend linearly on the abscissas of bar sections, and we obtain the graph (diagram) of displacements. It is seen from the graph that a certain section *D-D* does not displace. The sections above the section *D-D* move upward; the sections below move downward.

Example 2.2. Determine the elongation of a conical bar of circular cross section, if the smallest diameter is $2r_1$, and the greatest $2r_2$ (see Fig. 2.3).

Solution. The radius of a section of the bar at a distance z from the left end is

$$r_z=r_1+\frac{r_2-r_1}{l}z.$$

Hence the cross-sectional area of the bar at the distance z is

$$F_z = \pi r_z^2 = \pi \left(r_1 + \frac{r_2 - r_1}{l} z \right)^2 .$$

The elongation of the bar from formula (2.9) is

$$\Delta l = \frac{N}{\pi E} \int_0^l \frac{dz}{\left(r_1 + \frac{r_2 - r_1}{l} z \right)^2} = \frac{Nl}{\pi E r_1 r_2}$$

which at $r_1 = r_2 = r$ yields the elongation of a bar of constant (circular) cross section.

11. Experimental Study of the Properties of Materials

A. Purpose and Types of Tests

In order to study the properties of materials and establish the magnitude of allowable stresses specimens of the material are tested up to fracture. Experiments are carried out under loadings of the following types: static, impact and cyclic (fatigue or endurance test).

According to the type of strain produced in the specimen, tests may be classified as tension, compression, torsion and bending. Far less often tests are conducted under combined loading conditions, for instance, combined tension and torsion.

As experimental results depend on the form of a specimen, the rate of straining, testing temperature, etc., the experiment is usually carried out under standard conditions.

Tests are conducted with special machines of various designs and capacities. Special high-sensitivity instruments are employed for measuring deformations.

A detailed description of testing machines and instruments may be found in special textbooks *.

For static testing two identical specimens are taken as a minimum and for dynamic testing three specimens. Fatigue testing requires 6-8 identical specimens at the least.

---

* See, for instance, N. M. Belayev, *Laboratory Works in Strength of Materials.* Moscow, Gostekhizdat, 1954.

When less homogeneous materials are being tested the number of repeated tests should be increased, if possible, to obtain more reliable results.

B. Tensile and Compressive Test Diagrams

Tension and compression tests under static loading are most commonly used as they are simplest and at the same time in many cases they permit a fair prediction of the behaviour of a material under other types of loading.

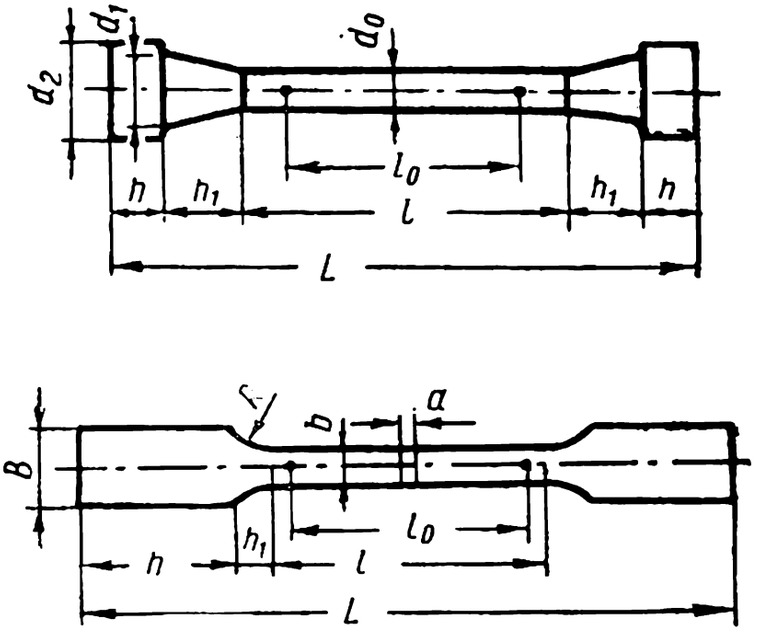


Fig. 2.7

Fig. 2.7 and Table 2.3 give data concerning some tensile test specimens used in the U.S.S.R. Specimens 20 mm in diameter are called *normal*, all others being called *proportional*.

The purpose of tension tests is to determine the mechanical characteristics of a material. During a test the relation between the force stretching the specimen (P) and the elongation of the specimen (Δl) is recorded automatically on a diagram.

So that test results may be compared for specimens of different dimensions made of the same materials, the tensile test diagram is reconstructed and represented in another system of co-ordinates; the magnitude of normal stress at a cross section of the tensile test specimen $\sigma = \frac{P}{F_0}$, where F_0 is the *original* cross-sectional area of

the specimen, is laid off along the ordinate axis and the unit elongations of the specimen $\varepsilon = \frac{\Delta l}{l_0}$, where l_0 is its *original* length, are laid off along the abscissa axis. This diagram is called the *conventional* tensile test diagram (or the diagram of conventional stresses) because stresses and unit elongations are calculated in relation to the *original* cross-sectional area and the *original* length of the specimen, respectively.

Table 2.3

Notation of specimen dimensions	Circular section		Rectangular section	
	tenfold specimen	fivefold specimen	tenfold specimen	fivefold specimen
L	$16\,d_0$	$10\,d_0$	$l+2(h+h_1)$	$l+2(h+h_1)$
h	d_0	d_0	$\approx 1.5\,b$	$\approx 1.5\,b$
h_1	$1.5\,d_0$	d_0	15 mm	15 mm
l	$11\,d_0$	$6\,d_0$	$\approx 13\sqrt{F_0}$	$\approx 6.5\sqrt{F_0}$
l_0	$10\,d_0$	$5\,d_0$	$11.3\sqrt{F_0}$	$5.65\sqrt{F_0}$
d_2	$\approx 1.8\,d_0$	$\approx 1.8\,d_0$	—	—
d_1	$\approx 1.2\,d_0$	$\approx 1.2\,d_0$	—	—
B	—	—	≈ 40 mm	≈ 40 mm
b	—	—	≈ 30 mm	≈ 30 mm
At d_0	from 5 to 25 mm	—	—	—
At a	—	—	from 4 to 25 mm	—

Fig. 2.8 gives a tensile test diagram for a low-carbon steel specimen with respect to co-ordinates ε, σ. As can be seen, in the beginning strains grow directly as stresses in the portion OA up to a certain stress σ_p, called *the proportional limit.* Consequently, Hooke's law holds up to the proportional limit. For steel Ст. 3 the proportional limit is $\sigma_p \approx 2{,}000$ kg/cm^2. With a further increase in the load the diagram becomes curved.

However, if stresses do not exceed a certain magnitude, *the elastic limit* σ_e, the material retains its elastic properties, i.e., after unloading the specimen recovers its original form and dimensions.

For steel Ст. 3 the elastic limit is $\sigma_e \approx 2{,}100$ kg/cm^2. The difference between the proportional limit and the elastic limit is not great and in practice no distinction is usually made between σ_p and σ_e.

If the load is increased still further, the moment is reached (point *C*) when deformations in fact begin to grow without the load being increased.

The horizontal portion *CD* of the diagram is called *the yield area.*

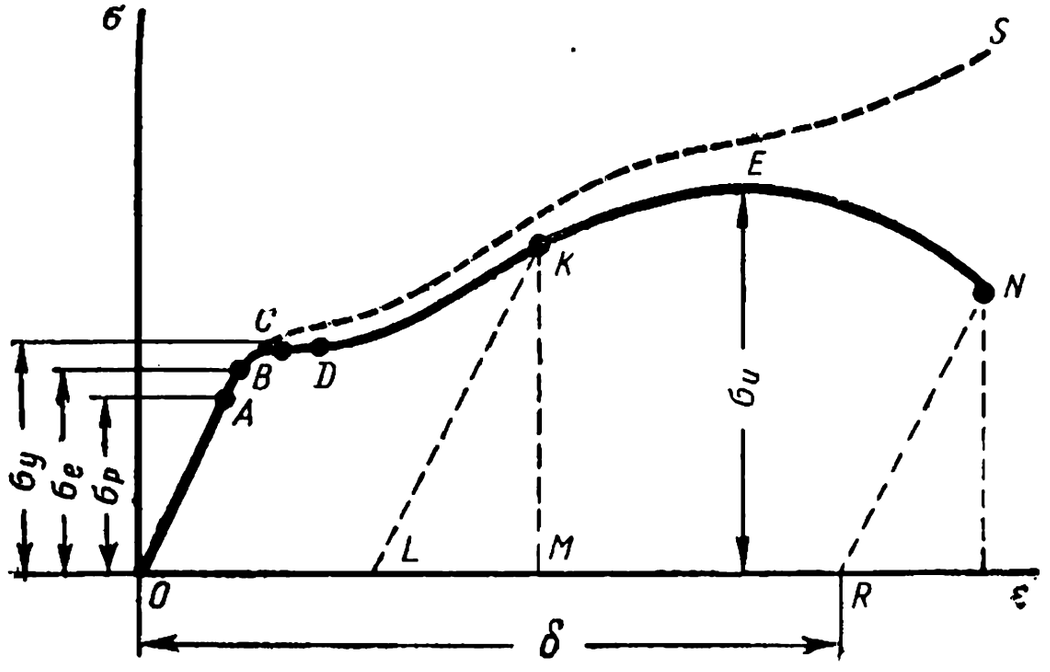


Fig. 2.8

The stress at which deformations grow with no increase in the load is called the yield point and is denoted by σ_y. *

For steel Ст. 3 the yield point $\sigma_y = 2{,}400$ kg/cm^2.

A number of materials in tension give a diagram without pronounced yield area; the so-called *yield strength* is established for these materials. *The yield strength* is the stress corresponding to a permanent set equal to 0.2 per cent. The yield strength is denoted by $\sigma_{0.2}$. Examples of materials for which the yield strength is determined are duralumin, bronze, high-carbon and alloy steels (for instance, for steel 37 ХН3А $\sigma_{0.2} = 10{,}000$ kg/cm^2).

---

* Sometimes this characteristic is called the physical yield point, to distinguish it from the yield strength which will be discussed below.

As investigations show, yielding is accompanied by considerable mutual sliding of steel crystals with the result that lines appear on the specimen surface (so-called *Chernov's lines* *) at an angle of about 45° to the axis of the specimen (Fig. 2.9*a*).

Having elongated to a certain extent under a constant value of force, i.e., having undergone a state of flow, the material regains its ability to resist extension (it hardens) and beyond point *D* the diagram rises, though far more gradually than before (see Fig. 2.8).

At point *E* the stress reaches its maximum value. This highest stress related to the original cross-sectional area of the specimen which the material can withstand is called the *ultimate (tensile) strength*. For steel Ст. 3 the ultimate strength is $\sigma_u = 4{,}000\text{-}5{,}000$ kg/cm^2. For high-strength

Fig. 2.9

steels the magnitude of the ultimate strength attains 17,000 kg/cm^2 (steel 40XHMA and others). The ultimate tensile strength is denoted by $\sigma_{u.t}$ and the ultimate compressive strength by $\sigma_{u.c}$.

Upon reaching the magnitude of ultimate strength a sharp local reduction of the cross section occurs, so-called *necking* (Fig. 2.9*b*). The cross-sectional area of the specimen at the neck reduces rapidly and, as a result, the force and conventional stress diminish. The rupture of the specimen takes place at the narrowest section of the neck.

In addition to the mechanical characteristics of the material listed above the *percentage elongation at rupture* δ is also determined in tension tests, which is an important characteristic of the ductility of the material:

$$\delta = \frac{l_1 - l_0}{l_0} 100\%, \qquad (2.14)$$

* After D. K. Chernov (1839-1921), the distinguished Russian metallurgist, who discovered the existence of phase transformations in steel and developed the theory of crystallisation of a steel ingot.

where l_0 is the original specimen length, l_1 is the specimen length after rupture.

For steel Ст. 3 $\delta \geqslant 24\%$. For high-strength steels this value is reduced to 7-10%. The magnitude of δ depends on the relationship between the length of the specimen and its lateral dimensions. Therefore the handbooks specify the kind of specimen for which the magnitude of δ was determined. For instance, δ_5 means that the elongation was determined with a fivefold specimen, i.e., a specimen in which the ratio of gauge length to diameter was five.

The elongation thus determined is an average elongation since the distribution of deformations is not uniform along the length of a specimen. The greatest elongation occurs in the zone of rupture. It is commonly termed the *true elongation at rupture.*

The second characteristic of the ductility of the material is the *percentage reduction in area of the cross section at rupture*:

$$\psi = \frac{F_0 - F_1}{F_0} 100\%, \qquad (2.15)$$

where F_0 is the original cross-sectional area, F_1 is the cross-sectional area at the narrowest place of the neck after rupture.

The quantity ψ characterises ductile properties more exactly than δ since it depends less on the shape of a specimen. For steel Ст. 3, the value of ψ is 50-60%. For most steels the magnitude of ψ is close to this value.

As has been pointed out above, tensile test diagrams for many grades of steel as well as alloys of non-ferrous metals have no yield area. The characteristic shape of the tensile test diagram for such materials is shown in Fig. 2.10.

To study considerable plastic deformations it is necessary to know the *true tensile test diagram* which gives the relation between true strains and true stresses defined as the tensile force divided by the actual cross-sectional area of the specimen (with proper account of the reduction in area).

Since the actual cross-sectional area is less than the original area, the diagram of true stresses lies above the diagram of conventional stresses, especially after necking begins when an abrupt reduction in cross-sectional area of the specimen takes place (curve *OCS* in Fig. 2.8).

It is common practice to apply approximate methods of plotting the diagram of true stresses which are set forth in complete courses on strength of materials *.

The tensile test diagram (Fig. 2.8) which has been under consideration is characteristic of the so-called *ductile materials*, i.e., materials capable of developing considerable permanent deformations (δ) without breaking.

The greater δ the more ductile is the material. A group of highly ductile materials comprises copper, aluminium, brass, low-carbon steel, etc.

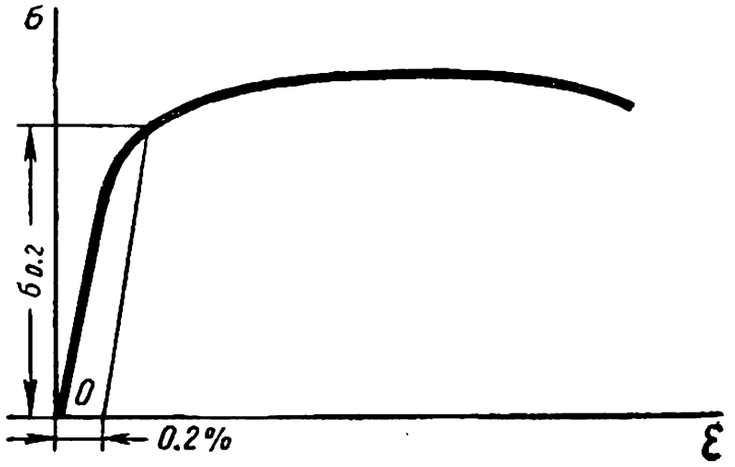


Fig. 2.10

Duralumin and bronze are less ductile.

Most alloy steels are slightly ductile materials.

A property contrary to ductility is *brittleness*, i.e., the ability of a material to fracture at small permanent deformations. For such materials the magnitude of the percentage elongation at rupture does not exceed 2 to 5 per cent and in some cases it is expressed by a fraction of per cent. Brittle materials comprise cast iron, high-carbon tool steel, stone, concrete, glass, fibre glass reinforced plastics, etc. It is to be noted that the division of materials into ductile and brittle ones is conditional since brittle materials may behave as ductile and ductile materials as brittle depending on the kind of stress state, and the conditions of testing (rate of loading, temperature, etc.).

For example, a cast-iron specimen behaves as a ductile material under conditions of uniform compression, i.e., no

---

* See V. I. Feodosyev, *Strength of Materials*. Moscow, Fizmatgiz, 1962.

fracture occurs even at considerable deformations. Conversely, a notched steel specimen will fracture at a relatively small deformation.

Thus it is more correct to speak of *ductile and brittle states of a material.*

In stretching specimens of brittle materials a number of peculiarities can be observed. The tensile test diagram of cast iron typical of all brittle materials is shown in Fig. 2.11. It is seen from the diagram that the deviation from Hooke's law begins very early. Fracture occurs suddenly with very small deformations and without necking.

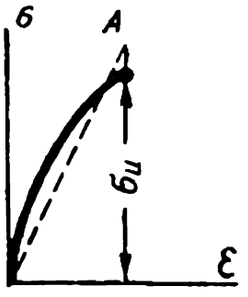


Fig. 2.11

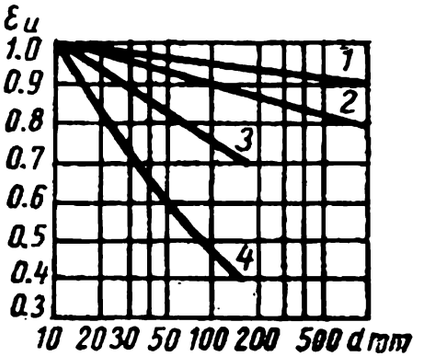


Fig. 2.12

This diagram makes it possible to determine two characteristics: the ultimate tensile strength σ_u and the percentage elongation at rupture. In practical calculations the deviation from Hooke's law is not usually taken into consideration, i.e., the curved diagram is replaced by the conventional straight diagram (see dashed line in Fig. 2.11).

For cast iron and other brittle materials the ultimate tensile strength is greatly influenced by specimen dimensions. This is evaluated by the size factor

$$\varepsilon_u = \frac{\sigma_{ud}}{\sigma_{u10}}. \qquad (2.16)$$

Here σ_{ud} is the ultimate strength of a specimen having a diameter d, σ_{u10} is the ultimate strength of a specimen having a diameter $d=10$ mm. Fig. 2.12 shows the curves of ε_u versus the specimen diameter for the following materials: (1) high-carbon and manganese steel, (2) alloy steel, (3) modified cast iron, (4) grey cast iron.

The magnitude of ε_u is particularly affected by the increase in absolute specimen dimensions for cast iron (see curves *3* and *4* in Fig. 2.12).

It is to be noted that in recent years a great success has been made in producing high-strength materials.

The theoretical value of the ultimate tensile strength calculated with due account of the interaction of atoms in a crystal is about one-tenth of E, i.e., about 200,000 kg/cm^2 for steel, which is about 10 times as great as the ultimate tensile strength for existing grades of high-strength steels.

In laboratory conditions filamentary iron crystals have already been obtained with the ultimate tensile strength $\sigma_u = 140{,}000$ kg/cm^2.

Methods have been found for strengthening steels and some alloys by neutron radiation. Rapid particles knock metal atoms out of the crystalline lattice, produce in it either free places, vacancies, or atoms without places, interstitials.

These distortions of structure (dislocations) make the metal more strong because they prevent moving within the crystal in the same manner as rough surfaces of two rods prevent their sliding.

C. Increase in the Yield Strength under Reloading (Strain Hardening)

If when loading a specimen the elastic limit is not exceeded, after unloading all deformations will completely disappear and on reloading this specimen will behave in the same way as in the first loading.

If, however, a specimen is loaded to a higher stress than the elastic limit, for example, to the stress corresponding to the point K on the diagram in Fig. 2.8, then reloading will follow the line KL parallel to the line OA. The elastic part of the deformation (portion LM) will disappear while the plastic part of the deformation (portion OL) will remain.

If the material is loaded again, the diagram will follow the straight line LK up to point K.* The permanent set

---

* Strictly speaking, the loading line does not quite coincide with the unloading line but the deviation is insignificant and can be neglected.

at rupture will be measured by the magnitude of the segment *LR*, i.e., it has a smaller magnitude than in the initial single loading to fracture.

Consequently, in reloading a specimen previously stretched to a higher stress than the yields point, the proportional limit is raised to the level reached by the stresses in the preceding loading. If there has been a pause between unloading and reloading, the proportional limit is raised still more.

It should be noted that the reloading diagram *LKEN* has no yield area, therefore for a specimen unloaded and reloaded the yield strength $\sigma_{0.2}$ is determined which is obviously higher than the yield strength in the initial loading. In the above sense it may be interpreted to mean an increase in the yield strength under reloading.

The phenomenon of increasing the proportional limit and reducing the ductility of a material under reloading is called *strain hardening*. In many cases strain hardening is an undesirable effect because the hardened metal becomes more brittle. However, in a number of cases strain hardening is useful and it is produced artificially, for instance, in parts subjected to repeated loadings (see Chapter XII).

D. Effect of Time on Deformation. Elastic After-Effect. Creep. Relaxation

Experiments show that deformation under loading does not occur instantaneously but develops over a certain period of time.

If at stresses corresponding to point *S* (Fig. 2.13) the loading is suspended and the specimen is allowed to remain for some time under the load, deformation will grow (segment *ST*) at first rapidly and then more slowly. After unloading a part of deformation corresponding to segment *GJ* will disappear almost instantly, whereas the other part of deformation represented by segment *OG* will require some time to disappear. *

This phenomenon of time-dependence of elastic deformations is termed *the elastic after-effect*. The more homogene-

---

* In Fig. 2.13 segment *ST* is represented on a considerably larger scale than the remainder of the diagram so that it can be seen more clearly.

ous the material, the less significant the elastic after-effect. For refractory materials it is so vanishingly small at normal temperatures that it may be neglected. On the other hand, in organic materials the elastic after-effect is great and cannot be ignored.

Another phenomenon is observed for many materials under loading at high temperatures—a continuous increase in permanent strains causing failure of the material under certain conditions.

For example, a steel steam pipe subjected to certain steam pressure and temperature continually increases in diameter.

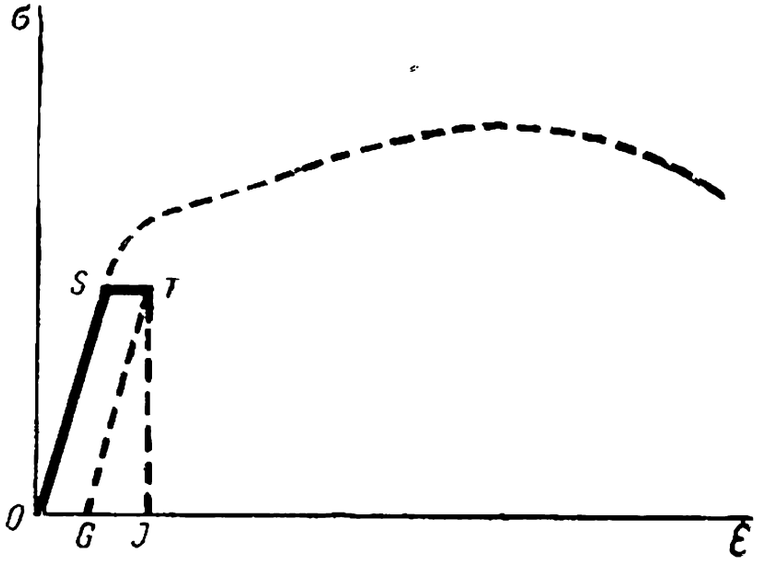


Fig. 2.13

The variation in strains with time in a loaded part is called *creep.*

For metals with a low melting point (for instance, zinc, lead) as well as for concrete, creep can be observed at normal temperatures. For steel appreciable creep occurs at temperatures exceeding 300° C.

The stress at which the strain rate at a given temperature and under a constant load is equal to a certain prescribed magnitude, for instance, 0.0001 per cent per hour, is called the *creep limit* σ_{cr} and is an important mechanical characteristic.

Creep is closely related to another phenomenon in which elastic deformations of a body become plastic with time and a consequent change in acting stresses takes place

with constant total deformation. This phenomenon is called *relaxation*. Due to relaxation, interference fit joints loosen under long service at high temperatures.

E. Temperature Effect

Experiments show that the properties of materials depend a good deal on temperature.

Fig. 2.14 presents several tensile test diagrams for low-carbon steel (0.15% C) at different temperatures and Figs. 2.15 and 2.16 present graphs of elastic constants

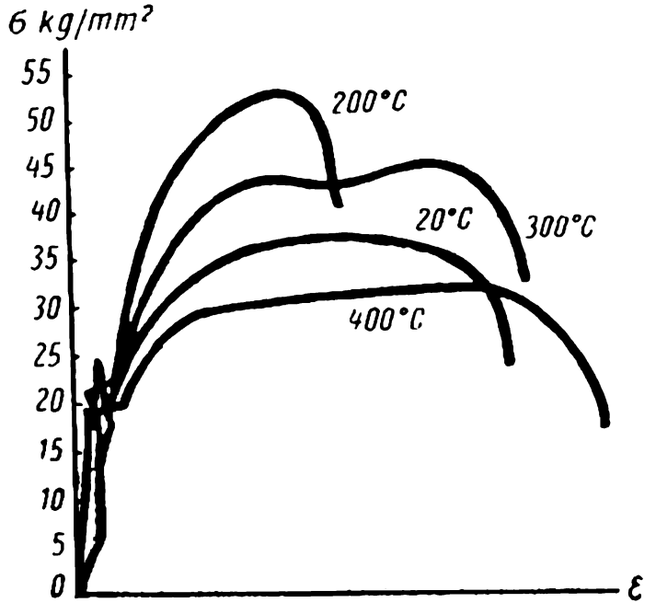


Fig. 2.14

(E and μ) and mechanical characteristics (σ_p, σ_y and σ_u) as well as ψ and δ versus temperature for the same steel.

The graphs show how greatly temperature affects the properties of steel. Up to 300° C the ultimate tensile strength σ_u increases (by 20-30%) and with a further rise in temperature it decreases sharply.

The magnitudes of the yield point σ_y and of the proportional limit σ_p decrease with an increase in temperature. At 400° the yield point is 60-70% of its magnitude at room temperature. When temperature is increased the length of the yield area is reduced and at a temperature of about 400° the area disappears altogether.

Ductile properties (percentage elongation at rupture δ and percentage reduction in area of the cross section ψ)

decrease with an increase in temperature up to 300° and with a further rise they increase (see Fig. 2.16).

The mechanical properties of materials depend very much on the duration of a test.

At some temperatures (for instance, for low-carbon steel at a temperature above 800°) the test piece may fail at a stress lower than the proportional limit at room temperature if this stress acts long enough. For this reason the strength of metals at high temperatures is characterised by so-called *stress-rupture strength* σ_{u_t} rather than by the usual ultimate tensile strength determined in short-time

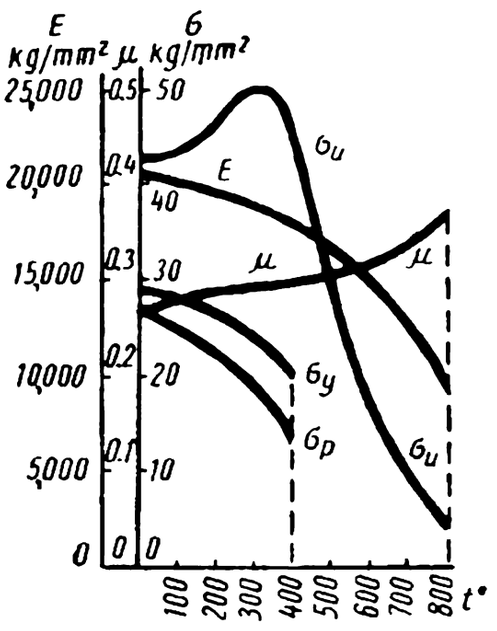


Fig. 2.15

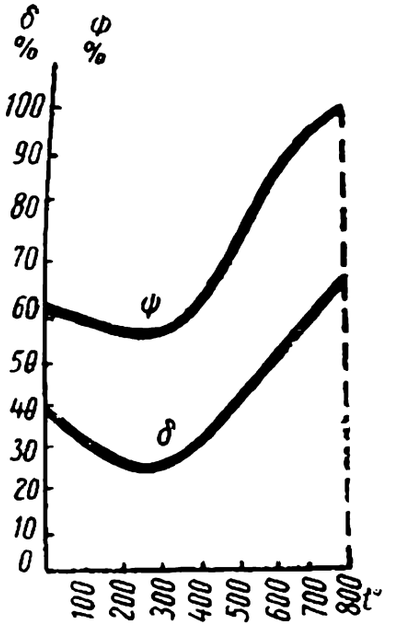


Fig. 2.16

tests. The stress-rupture strength is the stress the action of which over a certain period of time at a constant temperature leads to failure of the specimen.

Parts designed for service at high temperatures are made of special heat-resistant and heatproof steels containing some special alloying elements.

A *heat-resistant* steel is one that is able to maintain high strength at elevated temperatures, in particular, high resistance to creep. For increasing heat-resistance steel is alloyed with tungsten, molybdenum, vanadium.

A *heatproof* steel is one that is able to resist chemical surface damage due to hot air or gas (gas corrosion). To make steel more heatproof it is alloyed with chrome, silicon and aluminium.

F. Some Features of Compression Tests

For compression tests specimens are taken in the form of cubes or short cylinders with a height $h \leqslant 3d$, as bending may be involved for longer specimens. The use of very short specimens is also undesirable since friction forces occurring at the ends of the specimen prevent its expansion with the result that the specimen becomes barrel-shaped (Fig. 2.17*a*, *b*).

The results of a compression test are seriously affected by the magnitude of these friction forces. In this respect cylindrical specimens are preferable to cubical ones.

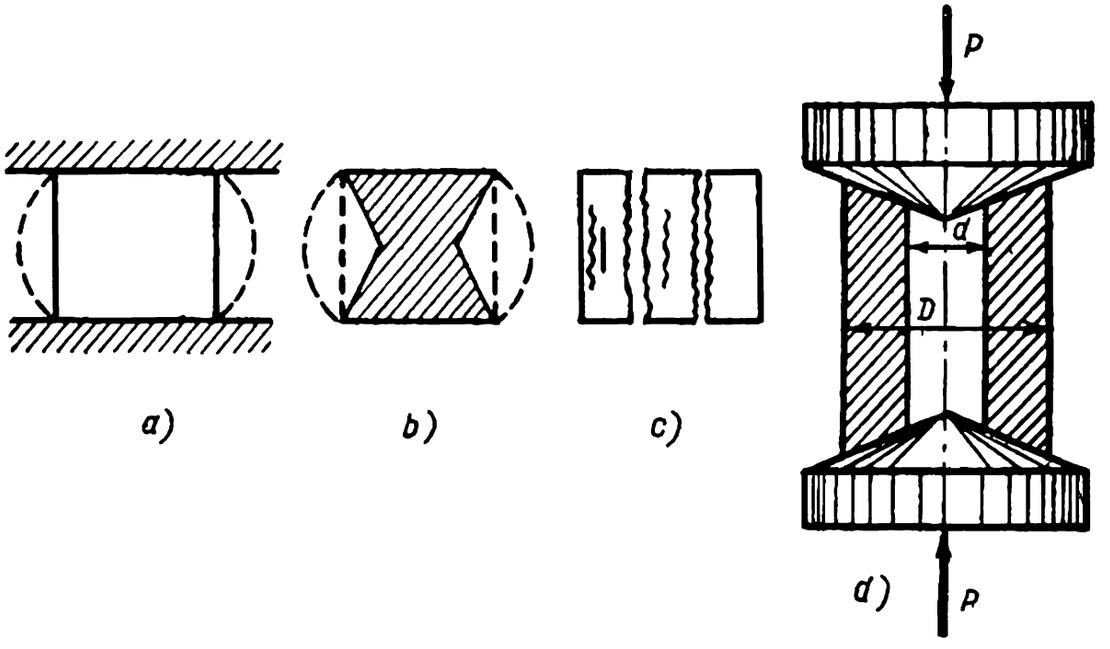


Fig. 2.17

The influence of friction forces can be partially eliminated by lubrication (for instance, by paraffin) of the specimen ends.

Fig. 2.17*b* shows the type of fracture of a stone cube in the absence of lubrication, and Fig. 2.17*c* in the presence of lubrication.

Lately hollow specimens with conical end surfaces (Fig. 2.17*d*) have been used for compression tests. By properly selecting the angle of the cone it is possible to reduce the influence of friction forces considerably.

The compression test diagram of a brittle material is analogous to its tensile test diagram (see Fig. 2.11). Fracture takes place with insignificant deformations. From this diagram it is possible to establish the ultimate compressive

strength σ_{uc} and the percentage permanent set at fracture δ.

For brittle materials the ultimate compressive strength σ_{uc} is much higher than the ultimate tensile strength σ_{ut}. A typical compression test diagram for a ductile material (low-carbon steel) is shown in Fig. 2.18*a*. At the initial portion it is analogous to the tensile test diagram. Then the curve rises steeply due to an increase in the cross-sectional area of the specimen and the hardening of the material. No fracture occurs. The specimen merely flattens

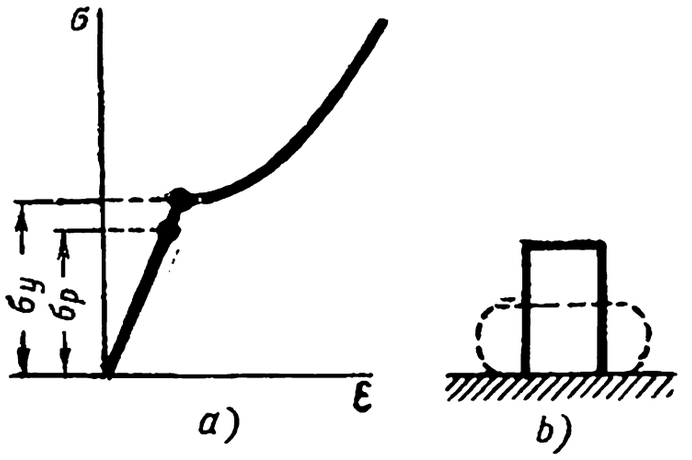


Fig. 2.18

out (Fig. 2.18*b*) and the experiment has to be stopped. The experiment gives the yield point in compression. For ductile materials the yield points in tension and in compression are practically the same but the yield area in compression is far less pronounced than in tension.

G. Mechanical Properties of Plastics

In recent years new materials based on natural and synthetic polymers, so-called *plastics*, have been more and more widely used in structures.

Plastics represent either pure resins or a composition of a resin and a series of components—filler, plasticiser, stabiliser, dye, etc.

Plastics are divided into composite and laminated ones according to the filler used. Composite plastics are in turn divided into powdered, fibrous and chips filled plastics.

Organic and inorganic *fillers* are used; they serve to modify the properties of a material, to improve the physico-

mechanical, frictional and other properties of a material as well as to reduce its cost.

Organic fillers are woodflour, cellulose, paper, cotton fabric. As inorganic fillers use is made of asbestos, graphite, glass fibre, glass cloth, mica, quartz and other materials.

Fillers in the form of cloth (woven or non-woven) make it possible to obtain laminated plastics of high strength.

With cotton cloth used as a filler a *cloth laminate* is obtained, with glass cloth a *glass cloth laminate*, with paper *hardened paper*, with asbestos cloth an *asbestos cloth laminate*, with wood veneer a *wood laminate.*

Reinforcing materials on a base of glass fibre, glass rope, glass mat constitute a special group of fillers which may ensure the manufacture of parts not yielding to steel in strength (Table 2.4). The data of Tables 2.4 and 2.5 relate to short-duration loading.

Table 2.4

Glass filler	Ultimate strength σ_u, kg/cm^2	Modulus of elasticity E, kg/cm^2
Glass mat	1,400-2,100	(80-120) 10^3
Diamond glass mat	5,000-6,000	(18-23) 10^4
Glass cloth	1,800-3,500	(14-21) 10^4
Parallel glass fibres	7,800-10,500	(23-40) 10^4

Distinctive features of plastics as structural materials are the following.

1. Stress-strain diagrams for plastics are linear to fracture or nearly so, and the elongation at rupture of most structural plastics does not exceed 2 to 3 per cent.

Fig. 2.19 presents tensile test diagrams for some plastics.

2. Plastics usually have different mechanical characteristics in tension and compression.

3. Plastics resist repeated and prolonged loadings much worse than metals.

4. Elastic and strength properties of plastics are characterised by larger amounts of scatter than for metals. This is due to strain-ageing of materials, hygroscopy, tempera-

ture effects, anisotropy of properties, non-homogeneity of structure, effects of manufacturing technology.

5. Plastics are characterised by a greater size effect as compared with metals. The strength of parts made of plastics decreases substantially with increase in dimensions of the cross section.

Fig. 2.20 presents the values of the size factor in tension for fibre glass reinforced plastics plotted in relation to the cross-sectional area of the specimen.

6. The properties of plastics depend a good deal on temperature. Fig. 2.21 presents graphs showing the relation between the ultimate tensile strength and temperature.

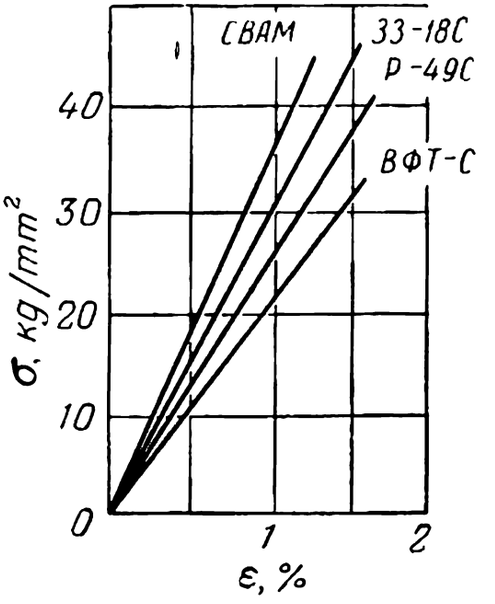


Fig. 2.19

The basic groups of plastics can operate in the temperature range from —60° C to +200°C; with the appearance of plastics based on organosilicon polymers and polyfluoroethylene resins the upper limit of temperature was raised to +500° C.

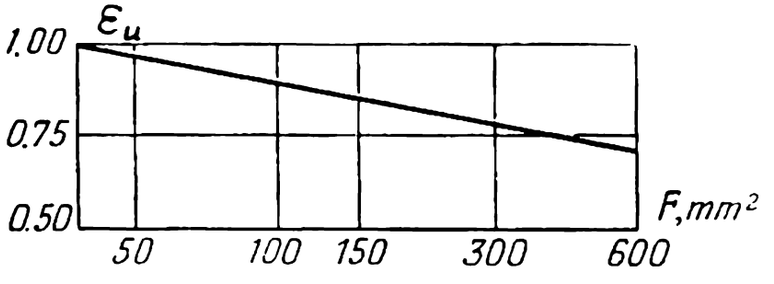


Fig. 2.20

7. Plastics have a greater tendency toward creep and relaxation even at normal temperatures.

8. Low rigidity; the modulus of elasticity of the most rigid plastics (fibre glass reinforced plastics) is about one-tenth of that for metals. In consequence parts made of plastics acquire larger deformations and displacements under load than metal parts.

9. In the design of laminated plastics the anisotropy of properties of a material must be taken into account by the methods of the theory of elasticity.

Table 2.5 gives information concerning the mechanical properties of the basic groups of plastics. It should be noted that much research work is still to be done on the study of the mechanical properties of plastics.

H. Non-Homogeneity of Materials

An assumption was made in Section 2 concerning the homogeneity and isotropy of the material a part is made of. However, in different parts made of one and the same

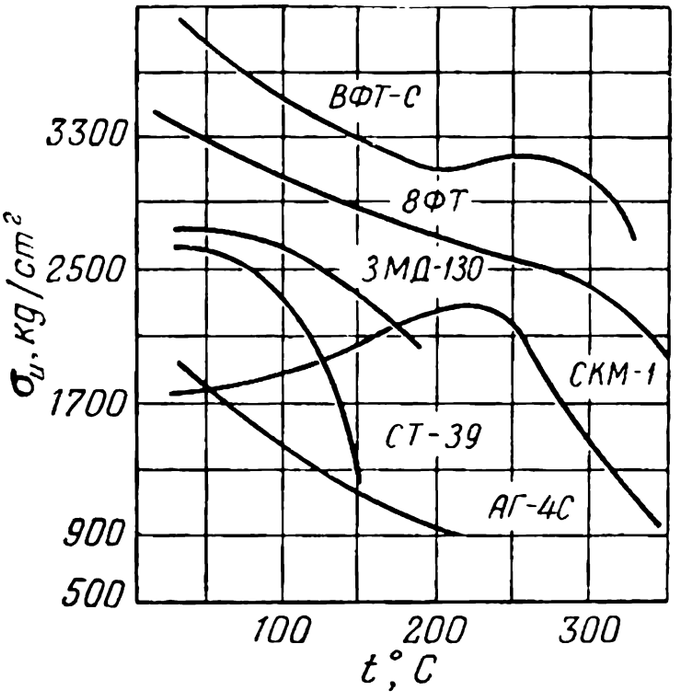


Fig. 2.21

material mechanical and ductile properties may differ a great deal.

If several identical specimens are made of one and the same steel and tested in tension to determine the yield point, the results will not coincide, as a rule.

By way of illustration Fig. 2.22 shows the yield point distribution curve for low-carbon steel Ст. 3 plotted according to experimental results for 6,000 specimens. The dashed line shows the test frequency diagram, the solid line represents the theoretical curve (the so-called normal distribution curve) to which the curve plotted according to experimental data approximates with an increase in the number of tests.

Table 2.5

Product	Basic material	Specific weight, g/cm³	Ultimate tensile strength, σ_{ut} kg/mm²	Ultimate compressive strength, σ_{uc} kg/mm²	Ultimate bending strength, σ_{ub} kg/mm²	Modulus of elasticity in tension, E_t kg/mm²	Shearing modulus, G kg/mm²	$\frac{\sigma_{-1}}{\sigma_{ut}}$
Fibre glass reinforced plastics	Fabric	1.4-1.85	26-40	10-30	13-45	1800-2200	350-400	0.22 0.25
	Biaxial-oriented fibres	1.7-1.9	30-50	—	23-46	2400-3500	—	0.25-0.28
Cloth laminates	Cotton cloth	1.3-1.4	6-11	13-15	9-16	950-1000	250	0.2-0.25
Wood laminates	Various sorts of wood	1.2-1.4	14-22	12-15.5	16.5-22	1200-3400	80-250	0.25-0.3
Hardened paper	Sulphate paper	1.3-1.4	7-10	—	8-14	1000-1800	80-250	0.2-0.3
Fibre	Special sorts of paper	1.1-1.25	3-13	11-13	4-10	500-800	—	0.25-0.3
Organic glass	Methacrylic acid polymers and copolymers	1.18	7.1-9.2	—	9.9-15.3	290-416	—	0.1-0.16

It can be seen from this curve that $\sigma_y = 3{,}000$ kg/cm² (15 per cent of all cases) is the most frequently encountered value of the yield point for this steel. The minimum value is close to 2,200 kg/cm² and the maximum one to

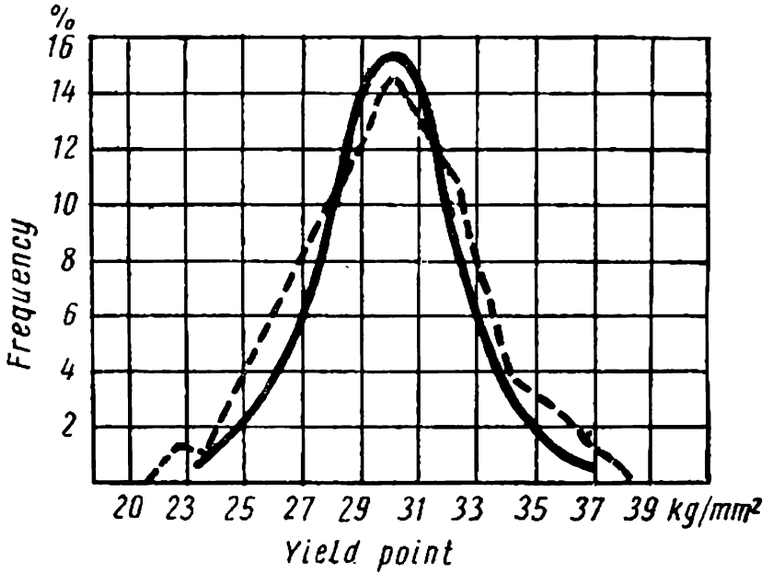


Fig. 2.22

3,800 kg/cm². The amplitude of yield point fluctuation is thus 800 kg/cm². The form of the curve indicates that it is not unlikely (over a greater number of tests) to observe, on the one hand, lower or, on the other hand, higher yield points.

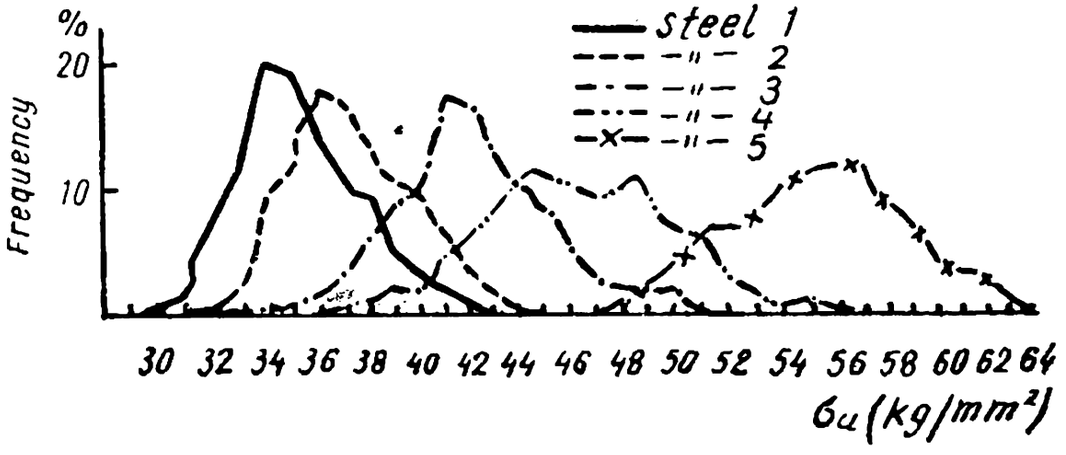


Fig. 2.23

Similar curves have been plotted for other materials. For less homogeneous materials, such as concrete and wood, they are not so steep, i.e., in this case the scattering (dispersal) of values of the corresponding quantity is considerably greater.

Fig. 2.23 shows frequency diagrams of ultimate strength for steels ranging from Ст. 1 to Ст. 5 from which it can be seen that steel Ст. 4 and steel Ст. 5 are less homogeneous than the other steels.

The availability of distribution curves giving a very reliable characteristic of homogeneity of a material permits parts to be designed more differentially, taking higher allowable stresses for more homogeneous materials, other things being equal.

Distribution curves underlie a principally new *statistical* method of structural analysis (see Sec. 114).

12. Factor of Safety. Choice of Allowable Stresses

We are aware now that both the actual loads acting on a part and the properties of materials from which it is made may differ a good deal, in an unfavourable sense, from those taken in design. Factors reducing the strength of a part (over-loads, non-homogeneity of materials, etc.) are of random character in most cases and cannot be estimated beforehand.

Since, however, parts and whole structures must operate safely under these unfavourable conditions as well, it is necessary to take certain precautionary measures. For this reason allowable stresses are assigned which are lower than the limiting stresses at which a part cannot continue to give normal service.

Thus, it is customary to take

$$[\sigma] = \frac{\sigma_{\text{lim}}}{n}, \tag{2.17}$$

where n is *the factor of safety*, σ_{lim} is the limiting stress of a material.

For brittle materials the ultimate strength is taken as the limiting stress, and for ductile materials the yield point, since at stresses equal to the yield point considerable plastic deformations occur which are not permissible.

Thus, the factor of safety is introduced to ensure that a structure and its individual parts are safe and reliable in exploitation in spite of a possible unfavourable deviation of actual service conditions from design ones.

The magnitude of n is decided on the basis of available service data of structures and machines.

Lately it has become the practice to divide the total safety factor n into a number of components, *partial factors of safety*, each reflecting the influence of a certain factor or group of factors on the strength of a structural element. For instance, one of these factors reflects possible deviations of the mechanical properties of a material from those accepted in design, another reflects deviations of the magnitudes of acting loads from their design values, and so on.

With such a breakdown of the total factor of safety it becomes possible to pay closer attention to the various concrete service conditions of machine parts and structures and so ensure greater reliability and economy of design.

The factor of safety is represented as the product

$$n = n_1 n_2 n_3 \ldots \tag{2.18}$$

As to the number of partial safety factors and their magnitude there is no uniformity as yet.

The values of the safety factors are usually based on experience in designing and operating machines of the type concerned. At present one, three, five and even ten partial safety factors are used in mechanical engineering. Designers are recommended to work with three partial factors of safety *

$$n = n_1 n_2 n_3 \tag{2.19}$$

where n_1 is the safety factor allowing for inaccuracy in determining loads and stresses. The value of this factor of safety may be taken as 1.2-1.5 when great accuracy is needed in determining the acting stresses and 2-3 when less accuracy is required;

n_2 is the safety factor allowing for non-homogeneity of a material, its high sensitivity to defects of mechanical treatment. The magnitude of n_2 in designing on the basis of the yield point under static loads may be taken from Table 2.6 (without considering the effect of absolute dimensions) as a function of the ratio between the yield point and the ultimate strength.

---

* Data concerning the use of partial factors of safety in the U. S. S. R. are taken from *Handbook of Mechanical Engineer.* Moscow, Mashgiz, Vol. 1-4, 1947—1948.

In designing on the basis of ultimate strength (for low-ductile and brittle materials) the magnitude of n_2 is taken as:

(a) for low-ductile materials (high-strength low-tempered steels) $n_2 = 2\text{-}3$;

(b) for brittle materials $n_2 = 3\text{-}4$;

(c) for highly brittle materials $n_2 = 4\text{-}6$.

Table 2.6

$\frac{\sigma_y}{\sigma_u}$	0.45-0.55	0.55-0.7	0.7-0.9
n_2	1.2-1.5	1.4-1.8	17.-2.2

In fatigue analysis (see Chapter XII) the magnitude of n_2 is taken as 1.5-2.0, being increased to 3 and more for low-homogeneity materials (especially for castings) and for parts of large dimensions;

Table 2.7

Materials	Allowable stresses, kg/cm²	
	in tension $[\sigma_t]$	in compression $[\sigma_c]$
Grey pig iron C4 12-84	200-300	700-1,100
Grey pig iron C4 15-24	250-400	900-1,500
Grey pig iron C4 21-40	350-550	1,600-2,000
Steel Ст. 0 and Steel Ст. 2	1,400	1,400
Steel Ст. 3	1,600	1,600
Steel Ст. 3 in bridges	1,400	1,400
Structural carbon steel in engineering	600-2,500	600-2,500
Structural alloy steel in engineering	1,000-4,000 and higher	1,000-4,000 and higher
Duralumin	800-1,500	800-1,500
Brass	700-1,400	700-1,400
Pine parallel to grain	70-100	100-120
Oak parallel to grain	90-130	130-150
Masonry	up to 2	6-25
Concrete	1-7.	10-90
Cloth laminate	300-400	300-400
Hardened paper	500-700	500-700

n_3 is the factor of service conditions which allows for the degree of responsibility of a part. Its magnitude is taken as 1-1.5.

In structural analysis three safety factors are employed: the factor of overloading, the factor of homogeneity and the factor of service conditions *. The values of these factors are given in specifications and standards for structural design.

Table 2.7 gives approximate values of allowable stresses for several materials under static loading.

13. Basic Types of Problems in the Design of Stretched (Compressed) Bars

After the stress at the most dangerous cross section of a stretched (compressed) bar has been determined on the basis of formula (2.2) and the allowable stress has been established in accordance with the consideration outlined above it is possible to make an estimate of the strength of the bar.

To ensure the strength of the bar it is necessary that actual stresses at its dangerous cross section should not exceed allowable stresses

$$\sigma = \frac{N}{F} \leqslant [\sigma]. \tag{2.20}$$

Here the allowable stress implies either tensile $[\sigma_t]$ or compressive $[\sigma_c]$ one, as the case may be.

Equation (2.20) is called the equation (or condition) of strength in tension (compression).

By the use of this equation the following problems can be solved:

1. *Verify the strength of a bar*, i.e., determine actual stresses on the basis of the given load and cross-sectional dimensions of the bar and compare them with allowable stresses. Actual stresses should not deviate from allowable stresses more than ± 5 per cent. Overstress above this value is not permissible from strength considerations and understress indicates over-expenditure of material.

---

* According to Structural Standards and Rules for Structural Design (1962) the factor of service conditions need not be specified separately.

2. *Determine* (on the basis of the known load and magnitude of allowable stress) *cross-sectional dimensions of a bar* required by the condition of its strength

$$F \geqslant \frac{N}{[\sigma]}. \tag{2.21}$$

3. *Determine the magnitude of the allowable longitudinal force* on the basis of the given cross-sectional dimensions of a bar and the known allowable stress

$$[N] \leqslant F[\sigma]. \tag{2.22}$$

After the allowable longitudinal force has been determined and the relation between the longitudinal force and load established (by means of the method of sections or in any other way) it is possible to find the allowable load.

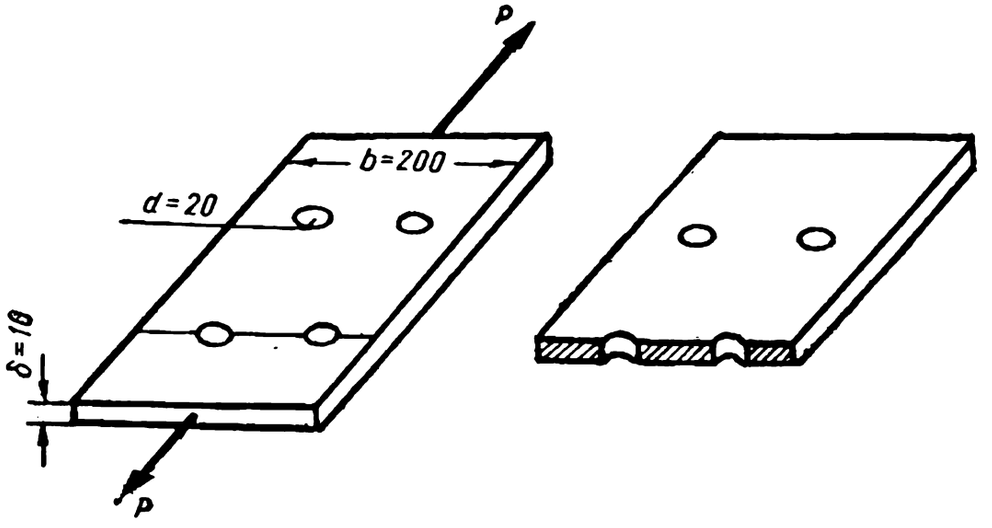


Fig. 2.24

It must be kept in mind that in addition to strength at the most weakened cross section *stability* also has to be considered in the design of compressed bars since buckling may occur at a certain value of the compressive force (see Chapter X).

Example 2.3. Determine the allowable load for a tensile steel sheet weakened by holes $d=20$ mm (Fig. 2.24). The allowable tensile stress $[\sigma_t]=1{,}600$ kg/cm². The sheet thickness $\delta=10$ mm, the width $b=200$ mm.

Solution. The allowable load is determined by an analysis of strength across the section weakened by the holes, as fracture is most likely to occur here. The total cross-sectional area of the sheet (F_{gross}) is

$$F_{\text{gr}} = 20 \times 1 = 20 \text{ cm}^2.$$

The weakening by the two holes is $\Delta F = 2 \times 1 \times 2 = 4$ cm^2. The working area of the section (F_{net}) is

$$F_{net} = F_{gr} - \Delta F = 20 - 4 = 16 \text{ cm}^2.$$

The allowable load is

$$[P] = F_{net} \times [\sigma_t] = 16 \times 1{,}600 = 25{,}600 \text{ kg}.$$

Example 2.4. Select the sections of the elements of the system represented in Fig. 2.25 and determine the displacement of the joint B. The material is Steel Ст. 3, $[\sigma] = 1{,}600$ kg/cm^2.

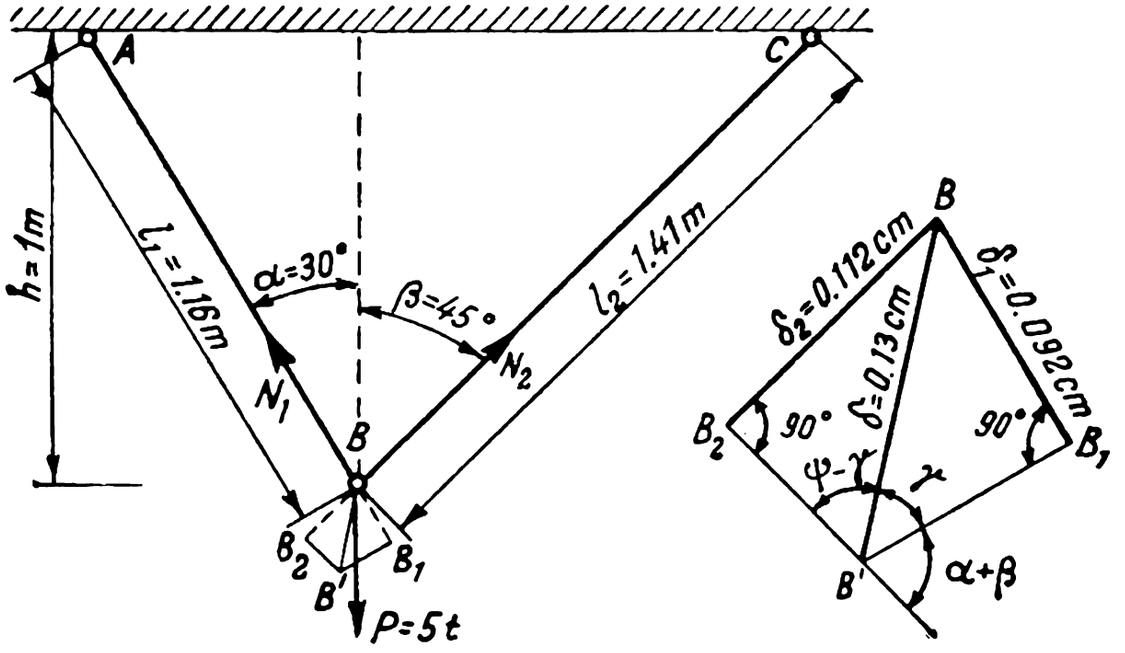


Fig. 2.25

Solution. Applying the method of sections, we determine the forces in the bars

$$\Sigma X = 0; \quad -N_1 \sin 30° + N_2 \sin 45° = 0, \text{ whence } N_1 = N_2 \sqrt{2};$$

$$\Sigma Y = 0; \quad N_1 \cos 30° + N_2 \cos 45° - 5 = 0.$$

Solving two equations with two unknowns, we find

$$N_1 = 3.69 \text{ t} \quad \text{and} \quad N_2 = 2.62 \text{ t}.$$

Both forces are tensile. We select the sections of the elements

$$F_1 = \frac{N_1}{[\sigma]} = \frac{3{,}680}{1{,}600} = 2.31 \text{ cm}^2,$$

$$F_2 = \frac{N_2}{[\sigma]} = \frac{2{,}600}{1{,}600} = 1.64 \text{ cm}^2.$$

Determine the displacement of the joint B. Find first the extensions of the bars

$$\delta_1 = \frac{N_1 l_1}{EF_1} = \frac{3{,}680 \cdot 116}{2 \cdot 10^6 \cdot 2.3} = 0.092 \text{ cm},$$

$$\delta_2 = \frac{2{,}600 \cdot 141}{2 \cdot 10^6 \cdot 1.62} = 0.112 \text{ cm}.$$

To determine the displacement of the joint B we apply the graphical method.

Let circular arcs of radii equal to the new lengths of the bars AB and BC (with proper account of the extensions) be swung from A and C as centres. They intersect at a point B', a new position of the hinge B. Because of the smallness of deformations the arcs may be replaced by straight lines B_2B' and B_1B' perpendicular to the directions of BC and AB. To improve the accuracy of the solution the *diagram of displacements* is plotted on an enlarged scale (Fig. 2.25*b*). The segment BB' represents the displacement of the joint B to the scale adopted.

In the present case the displacement is $\delta = 0.13$ cm.

As an alternative line of approach the analytic method may be employed. To do this it is necessary to apply twice the theorem of sines, to the triangles BB_2B' and BB_1B'. We obtain

$$\frac{\delta_1}{\sin \gamma} = \frac{\delta}{\sin 90^\circ} = \delta, \qquad \text{(I)}$$

$$\frac{\delta_2}{\sin (\psi - \gamma)} = \frac{\delta}{\sin 90^\circ} = \delta, \qquad \text{(II)}$$

where $\psi = 180 - \alpha - \beta = 105^\circ$. Equating the left-hand members of equalities (I) and (II) and using the trigonometric formula

$$\sin (\psi - \gamma) = \sin \psi \cos \gamma - \cos \psi \sin \gamma,$$

we obtain after transformations

$$\tan \gamma = \frac{\delta_1 \sin \psi}{\delta_2 + \delta_1 \cos \psi}.$$

Substitute the numerical data

$$\tan \gamma = \frac{0.092 \sin 105^\circ}{0.112 + 0.092 \cos 105^\circ} = \frac{0.092 \cdot 0.966}{0.112 - 0.092 \cdot 0.259} = 1.008,$$

$$\gamma = 45^\circ 16'; \quad \sin \gamma = 0.71.$$

From equality (I) or (II) we have

$$\delta = \frac{\delta_1}{\sin\gamma} = \frac{\delta_2}{\sin(105^\circ - \gamma)} = \frac{0.092}{0.71} = \frac{0.112}{0.864} = 0{,}129 \text{ cm}.$$

14. Statically Indeterminate Problems In Tension and Compression

For many systems composed of separate bars the forces in the latter cannot be determined from the equations of equilibrium alone. Such systems are called *statically indeterminate* systems.

As an example, consider the system represented in Fig. 2.26. Applying the method of sections we find three

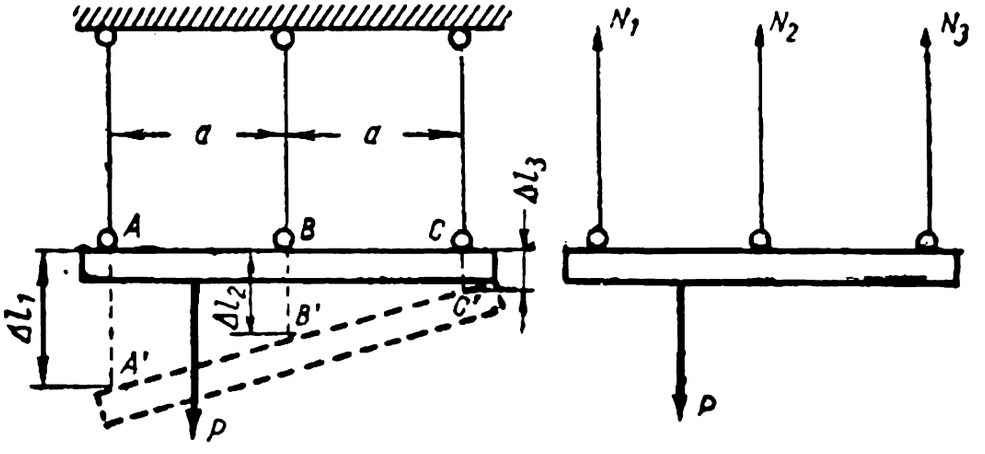


Fig. 2.26

unknown forces N_1, N_2 and N_3. However, we can derive only two equations of equilibrium: the sum of the projections on the vertical axis of all forces acting upon the cut portion must be equal to zero and the sum of the moments of these forces with respect to some point must be zero. The three unknowns cannot be determined from these two equations. It is necessary, therefore, to derive an additional equation which must express the condition of compatibility (continuity) of deformations of the members of the system. *In order to derive the equation of compatibility of deformations it is necessary to represent the system in a deformed state and to establish the relation between the deformations of different bars (members) of the system directly from the drawing (geometrically).* Thus, in the case under consideration, the beam AC which is supposed to be

perfectly rigid, will assume the position $A'C'$, shown by the dashed line, after the load is applied.

It follows from the drawing that $BB' = \Delta l_2 = \frac{\Delta l_1 + \Delta l_3}{2}$ (as the medium line of a trapezoid). Expressing now the elongations in terms of forces and stiffnesses of the bars (see formula 2.4) we shall obtain a third equation, after which we shall determine N_1, N_2 and N_3.

Example 2.5. Determine the allowable load for the system of steel bars shown in Fig. 2.27. The beam AC is assumed to be perfectly rigid. The allowable stress is $[\sigma]$.

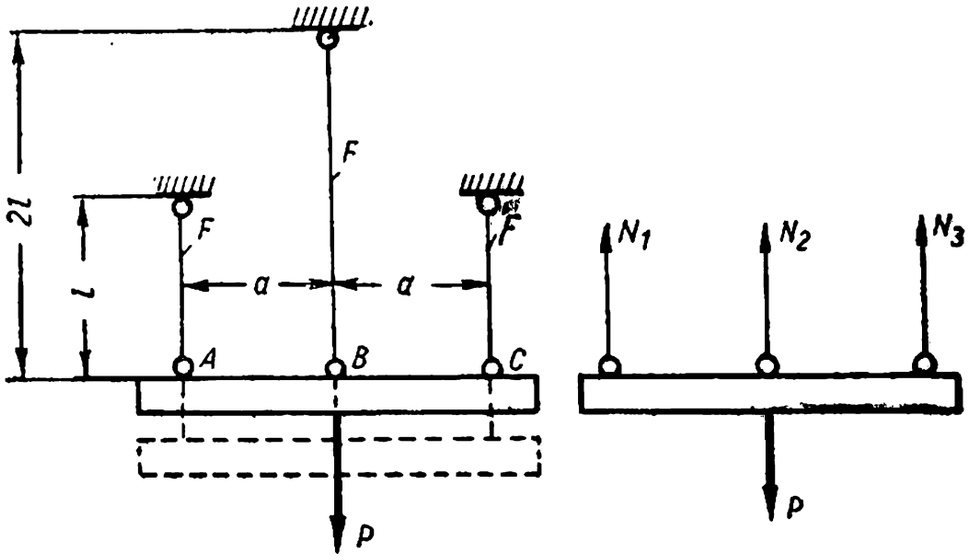


Fig. 2.27

Solution. To determine the forces in the bars we apply the method of sections. Derive the equations of equilibrium of the beam acted upon by a system of parallel forces

$$(1)\quad \Sigma Y = 0, \quad N_1 + N_2 + N_3 - P = 0,$$
$$(2)\quad \Sigma M_B = 0, \quad N_1 = N_3.$$

To derive the equation of deformations let us represent the system in a deformed state.

From the symmetry condition the beam AC will move downward, remaining parallel to its initial position. Therefore, the elongations of all the bars will be the same

$$\Delta l_1 = \Delta l_2 = \Delta l_3.$$

This is the equation of deformations. According to Hooke's law we have

$$\frac{N_1 l}{EF} = \frac{N_2 2l}{EF}, \quad \text{hence} \quad N_2 = \frac{N_1}{2}.$$

Substituting the values of N_2 and N_3 in the first equation we obtain

$$N_1 = N_3 = 0.4P, \qquad N_2 = 0.2P.$$

From the first equality we find $P' = 2.5N_1 = 2.5N_3$.
From the second equality we find $P'' = 5N_2$.

The allowable values of P will be found by substituting in these expressions the values of allowable longitudinal forces $[N_1] = [N_2] = [N_3] = F[\sigma]$.

We obtain

$$P' = 2.5\,[\sigma]\,F, \qquad P'' = 5\,[\sigma]\,F.$$

It is evident that of the two values of P the lesser must be taken as the allowable load, i.e.

$$[P] = 2.5\,[\sigma]\,F.$$

Thus, in this example the allowable load is limited by the strength of the outer bars.

In many cases for analysing statically indeterminate systems it is more convenient to employ another procedure. The idea of this will be illustrated by a concrete example. Let it be required to determine internal forces caused by a load in the bar represented in Fig. 2.28.

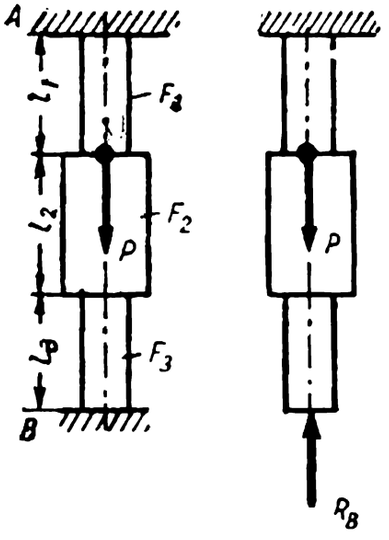


Fig. 2.28

Load P is carried partially by the upper fixed end and partially by the lower fixed end.

Two reactions arising at the built-in ends cannot be determined from a single equilibrium equation: the sum of the projections of all forces on the vertical axis must be equal to zero.

To derive the equation of deformations let us proceed as follows. Remove one fixed end, for instance, the lower one replacing its action upon the bar by an unknown reaction R_B.

In the system thus obtained (it is usually called *the basic system*) let us equate to zero the displacement of the lower section, as this section is fixed and cannot become displaced in the given system. Due to the action of force R_B section B will move upward as a result of the shortening of the whole bar, and due to the action of force P it will

move downward as a result of the elongation of the upper portion of the bar of length l_1 since force P with the lower fixed end removed is transmitted to the upper fixed end by means of this portion of the bar only.

To determine the elongations use is made of Hooke's law. As the cross-sectional area of the bar is not the same throughout, the deformation is determined in parts:

$$-\frac{R_B l_3}{EF_3}-\frac{R_B l_2}{EF_2}-\frac{R_B l_1}{EF_1}+\frac{Pl_1}{EF_1}=0.$$

Hence R_B is determined after which the longitudinal forces acting at the sections can readily be found by the method of sections as shown in the preceding paragraphs.

Example 2.6. Determine the forces acting in the bars of the system represented in Fig. 2.29. All bars have the same moduli of elasticity and the same cross-sectional areas.

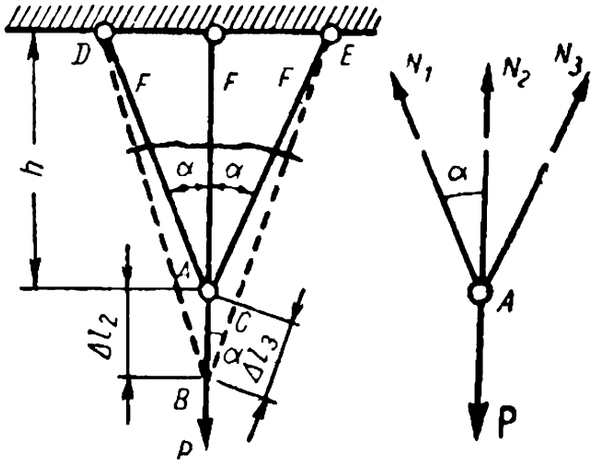


Fig. 2.29

Solution. Imagine joint A being cut away. To determine three unknown forces, in our case a system of concurrent forces, statics give only two equations of equilibrium. Consequently, the problem is once statically indeterminate.

To determine the forces consider the equilibrium of joint A. Equating to zero the sum of the projections of the acting forces on the horizontal axis, we obtain

$$(1)\ \Sigma X=0;\quad -N_1\sin\alpha+N_3\sin\alpha=0\ \text{ or }\ N_1=N_3.$$

This result also follows from the symmetry of the system. Equating to zero the sum of the projections of the acting forces on the vertical axis, we have

$$(2)\ \Sigma Y=0;\quad N_1\cos\alpha-P+N_2+N_3\cos\alpha=0.$$

By using the equality $N_1=N_3$ we obtain

$$2N_1\cos\alpha+N_2-P=0.$$

In order to derive the missing equation consider the deformations of the bars of the system.

Joint A will move to point B as a result of deformation of the bars. Segment AB represents elongation Δl_2 of the centre bar. Segment BC represents elongation Δl_3 of the right-hand bar. This segment will be obtained by laying off the length of the right-hand bar on line BE by means of compasses from point E. Since the deformations are small the circular arc AC may be replaced by a straight segment AC, a perpendicular from point A on the direction BE.

From the triangle ABC it follows that

$$\Delta l_3 = \Delta l_2 \cos \alpha.$$

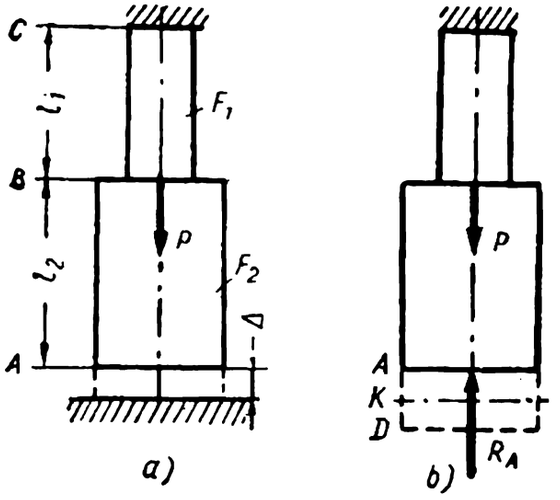


Fig. 2.30

Expressing the deformations in terms of forces (by use of Hooke's law), we obtain

$$\frac{N_3 l_3}{EF} = \frac{N_2 l_2}{EF} \cos \alpha.$$

But

$$l_2 = h; \quad l_3 = l_1 = \frac{h}{\cos \alpha}.$$

Taking this into account we have

$$N_3 = N_1 = N_2 \cos^2 \alpha.$$

Substituting this expression in the second equation of equilibrium ($\Sigma Y = 0$) we find

$$N_2 = \frac{P}{2 \cos^3 \alpha + 1};$$

$$N_1 = N_3 = \frac{P \cos^2 \alpha}{2 \cos^3 \alpha + 1}.$$

Consequently, the greatest force will act in the centre bar, and the strength of this bar will limit the allowable load for the system.

Example 2.7. Determine the reaction produced by the lower support on the bar represented in Fig. 2.30*a*.

Solution. There is a small clearance of magnitude Δ between the lower end of the bar and the support before the load is applied. Due to the action of the force the clearance will close up and reaction R_A will arise. To determine this reaction let us remove the lower support, replacing its action upon the bar by force R_A (Fig. 2.30*b*). Derive the equation of deformations. Segment AD represents the elongation of the bar which would result from the action of

force P in the absence of the support. Segment KD represents the contraction of the bar under the action of reaction R_A.

It is seen from the drawing that

$$\overline{AD}=\overline{KD}+\Delta \quad \text{but} \quad \overline{AD}=\frac{Pl_1}{EF_1}; \quad \overline{KD}=\frac{R_A l_2}{EF_2}+\frac{R_A l_1}{EF_1}.$$

Therefore

$$\frac{Pl_1}{EF_1}-\frac{R_A l_2}{EF_2}-\frac{R_A l_1}{EF_1}=\Delta.$$

This equation determines R_A. If R_A comes out positive, it means that we have assumed the right direction for R_A, i.e., upwards. If R_A has a negative value, it means that force P is not sufficient for the clearance to close up and it is necessary to apply to the lower end of the bar the downward force R_A.

Concequently, for a negative value of R_A the problem is reduced to determining the forces in the cross sections of the bar under the action of a single force P (statically determinate problem).

15. Thermal and Assembly Stresses. Initial Control of Forces in Structures

Consider two bars of which one (Fig. 2.31*a*) represents a statically determinate system, and the other (Fig. 2.31*b*) a statically indeterminate system.

With an increase in temperature of Δt a bar fixed at one end will have its lateral and longitudinal dimensions increased (Fig. 2.31*a*). The increase in length Δl_t, according to a formula known from physics, is $\Delta l_t=\alpha\Delta tl$, where α is the coefficient of thermal expansion; for instance, for steel $\alpha=125\times10^{-7}$.

Since there are no constraints preventing the bar from expanding, no internal forces will be induced in it.

When the temperature of a bar fixed at both ends (Fig. 2.31*b*) is raised by Δt, an internal compressive force will be induced as the second fixed end prevents the bar from expanding.

A general rule follows from this: *in statically determinate systems a change in temperature causes deformations without producing internal forces, in statically inde-*

terminate systems a change in temperature leads to the appearance of internal forces. To determine the latter let us apply the usual procedure for analysing statically indeterminate systems. Imagine one of the fixed ends, for instance the right one, being removed. Then the bar will be

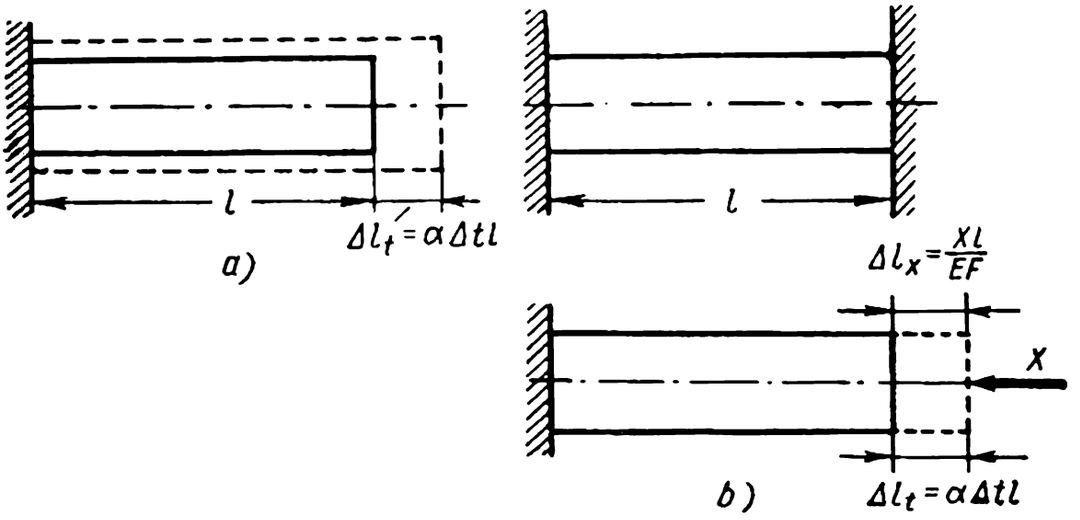


Fig. 2.31

free to expand by an amount $\Delta l_t = \alpha \Delta t l$. But the reaction X compresses the bar to the extent

$$\Delta l_x = \frac{Xl}{EF}.$$

The actual displacement of the right-hand end section is equal to zero, hence

$$\alpha \Delta t l = \frac{Xl}{EF}.$$

and, finally,

$$X = EF\alpha \Delta t \quad \text{or} \quad \sigma_t = \frac{X}{F} = E\alpha \Delta t. \qquad (2.23)$$

Thermal stresses σ_t may have very great values. To reduce them special *temperature clearances* (joints) are provided in structures.

In addition to stresses due to temperature effects, there may arise stresses in statically indeterminate systems when mounting a structure since individual bars of the structure may have departures from the design length on account of the inaccuracy of manufacturing.

Consider, for instance, the system represented in Fig. 2.32. Suppose that the bars are made of the same ma-

terial and have the same cross-sectional area. The distances between the bars are also the same, i.e., $AB=BC$. Let the length of the middle bar be less than is required by the geometrical scheme of the structure by the amount δ. When assembling the system the middle bar has to be stretched in one way or another to be able to join it to the beam ABC (for instance to weld). In consequence there arise some forces in the bars of the system after mounting.

To determine these forces we use the method of analysis of statically indeterminate systems which is already familiar to us. Applying the method of sections, we obtain from the conditions of equilibrium

1. $\Sigma M_B=0$, whence $N_1=N_3$.
2. $\Sigma Y=0$, whence $N_2=2N_1$.

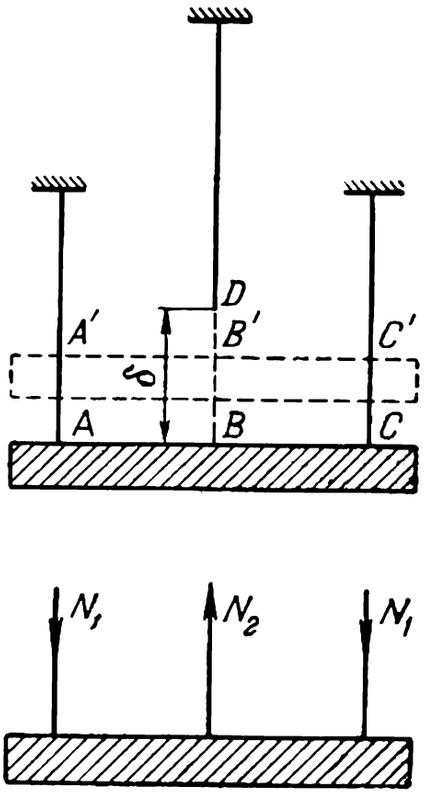


Fig. 2.32

From the condition of compatibility we obtain

$$\overline{AA'}=\overline{BB'}=\overline{DB}-\overline{DB'}.$$

But

$$\overline{AA'}=\Delta l_1,$$

$$\overline{DB}=\delta,$$

$$DB'=\Delta l_2.$$

Hence

$$\Delta l_1=\delta-\Delta l_2$$

or

$$\frac{N_1 l}{EF}=\delta-\frac{N_2 2l}{EF}.$$

Substituting here the value $N_2=2N_1$, we obtain

$$N_1=\frac{\delta EF}{5l} \quad \text{and} \quad N_2=\frac{2}{5}\frac{\delta EF}{l}.$$

The positive sign in front of N_1 and N_2 shows that our assumptions of their directions are correct, i.e., the extreme bars are compressed, and the middle bar is stretched.

If a load is now applied to this system, say, a force P at the point B, the forces in all the bars will be tensile and given by (see Example 2.5 and Fig. 2.27)

$$N_1 = N_3 = \frac{2}{5}P; \quad N_2 = \frac{1}{5}P.$$

Summing up these forces with the assembly forces, we obtain

$$N_1 = \frac{2}{5}P - \frac{\delta EF}{5l},$$

$$N_2 = \frac{P}{5} + \frac{2}{5}\frac{\delta EF}{l}.$$

By varying the amount of the clearance δ we can suitably adjust forces and stresses in the bars of a statically indeterminate system.

The magnitude of δ may be chosen so as to make, say, the stresses in all the bars of the system the same. Equating $\sigma_1 = \frac{N_1}{F} = \sigma_2 = \frac{N_2}{F}$, we obtain the value $\delta = \frac{Pl}{3EF}$ for δ. The stresses in all the bars are then the same and given by

$$\sigma_1 = \sigma_2 = \sigma_3 = \frac{5}{15}\frac{P}{F} = \frac{1}{3}\frac{P}{F}.$$

In the system without force control (Fig. 2.27) we would have

the stress in the extreme bars

$$\sigma_1 = \sigma_3 = \frac{2}{5}\frac{P}{F} = 0.4\frac{P}{F},$$

the stress in the middle bar

$$\sigma_2 = \frac{P}{5F} = 0.2\frac{P}{F}.$$

Comparing these results, we see that the initial stretching of the middle bar has permitted the design stress to be reduced from $0.4\frac{P}{F}$ to $0.33\frac{P}{F}$.

In recent years the idea of initial control of forces has found a wide application in various structures. It

has found an especially extensive application in reinforced concrete construction (prestressed reinforced concrete).

The reinforcement stretched before concreting is released from stretching devices after placing concrete and its setting and produces a state of stress in the reinforced concrete element reverse to that which will be developed in it by the load. Of special significance is the reduction of tensile stresses in concrete since concrete resists weakly the action of tensile stresses.

At present a mass production of prestressed concrete structures is effected in the U.S.S.R. (and in other countries of the world).

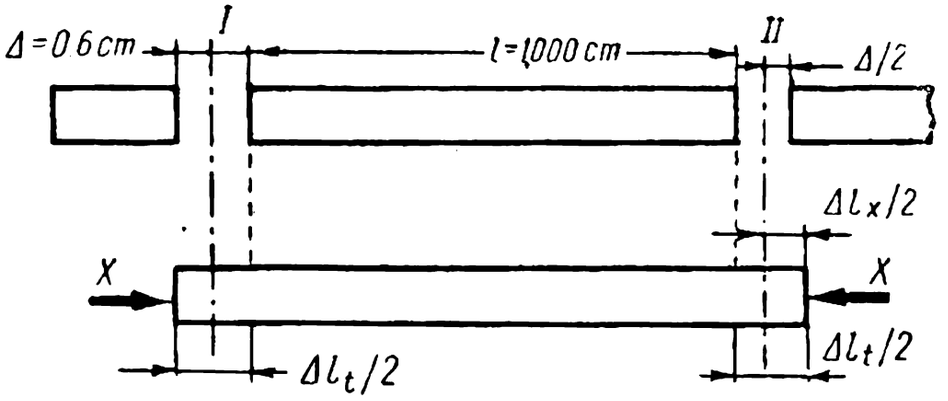


Fig. 2.33

Example 2.8. Determine stresses arising in steel rails (tram or railway lines) in summer at $t=+30°$ C if the rails have been laid without clearances in winter at $t=-30°$ C. For steel $\alpha=125\times10^{-7}$ and $E=2\times10^{6}$ kg/cm^2.

Note. When the international system of units (Si) is used the temperature is measured in degrees Kelvin. It is then necessary to write $t=273+30=303°$ K in place of $t=+30°$ C, and $t=273-30=243°$ K in place of $t=-30°$ C.

Solution. According to formula (2.23) we have for $\Delta t=60°$, $\sigma_t=125\times10^{-7}\times2\times10^{6}\times60=1{,}500$ kg/cm^2, which is close to allowable stresses for low-carbon steel.

To reduce thermal stresses in the rails clearances are left at certain intervals along the track, which is very undesirable from other points of view.

Example 2.8a. Determine the stresses in the same rail if clearances $\Delta=6$ mm are provided every 10 m (Fig. 2.33).

If there were no restraints exerted by adjacent rails, the elongation of the rail concerned resulting from temperature rise would be $\Delta l_t=\alpha l\Delta t$.

The reactive force X, which is induced after the clearance has closed up (when the length of the rail equals $l+\Delta$, i.e., the distance between lines I and II), compresses the rail by the amount

$$\Delta l_x = \frac{Xl}{EF} = \frac{\sigma_t l}{E}.$$

It is seen from the drawing that $\Delta l_t = \Delta l_x + \Delta$. Substituting here the values of Δl_t, Δl_x and Δ we obtain: $\sigma_t = 300$ kg/cm^2. Consequently, a small clearance reduced the thermal stresses considerably. It can easily be proved that with a clearance of 7.5 mm the stresses σ_t will be equal to zero.

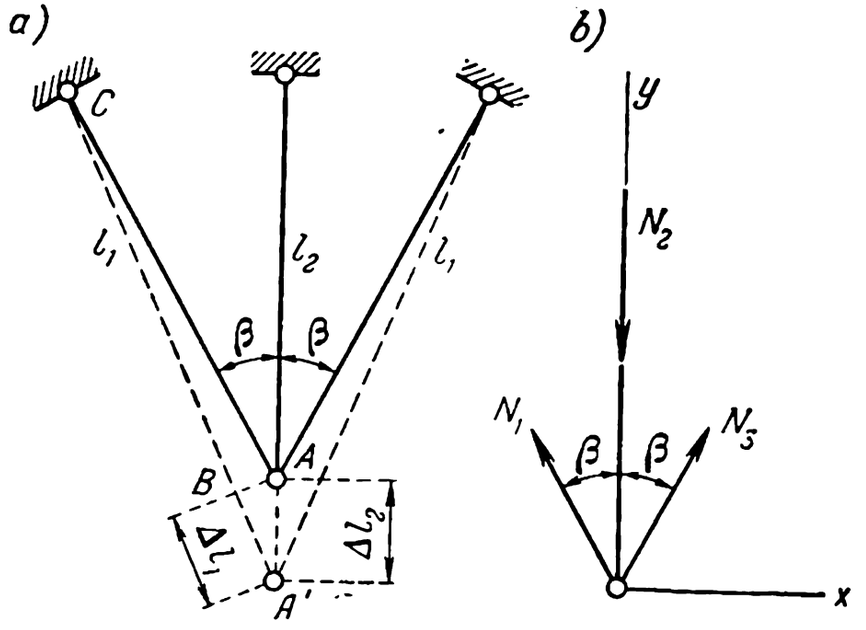


Fig. 2.34

Example 2.9. In a system of bars of the same material and cross-sectional area (Fig. 2.34*a*) the centre bar is heated by Δt. Determine the forces in the bars.

Solution. We apply the method of sections and derive the equations of equilibrium for the cut portion (Fig. 2.34*b*):

$$(1)\ \Sigma X = 0, \qquad N_1 = N_3,$$

$$(2)\ \Sigma Y = 0, \qquad 2N_1 \cos\beta - N_2 = 0.$$

To derive the equation of deformations we represent the system in a deformed state. Point A will assume position A' due to deformation. Segment AA' gives the elongation of the centre bar. To determine the elongation of

the left-hand bar we drop a perpendicular from point A on direction $A'C$, obtaining a point B. Segment $A'B$ represents elongation Δl_1 of the inclined bar. Since the deformations are small the angle at vertex A' of triangle $A'BA$ may be considered equal to the initial angle.

Then from triangle $A'BA$ we have

$$\Delta l_1 = \Delta l_2 \cos \beta.$$

This is the equation of deformations. It remains to express the deformations in it in terms of forces and to take into consideration the deformation of the centre bar due to temperature:

$$\frac{N_1 l_1}{EF} = \left(\alpha l_2 \Delta t - \frac{N_2 l_2}{EF}\right) \cos \beta.$$

By solving the system of three equations with three unknowns we determine N_1, N_2 and N_3.

16. Stresses on Inclined Sections under Tension (Compression) in one Direction

In order to gain a clear insight into the strength of a material it is necessary to be able to determine stresses acting on any inclined section of a stretched (compressed) element (Fig. 2.35). The normal stresses at the cross section of a bar σ_1 are considered to be known $\left(\sigma_1 = \frac{N}{F}\right)$.

Let us determine the stresses arising at the inclined section AB the normal to which makes an angle α with the direction of σ_1. The positive direction of angle α will be taken in a counter-clockwise direction.

Let F denote the area of the cross section perpendicular to the axis of the bar and let F_α denote the area of the inclined section; then

$$F_\alpha = \frac{F}{\cos \alpha}. \qquad (2.24)$$

In the general case both normal stresses σ_α and shearing stresses τ_α may act on the inclined section. Their magnitude will be found from the condition of equilibrium of the cut, e.g., the lower portion (Fig. 2.35*b*). Project the forces on the direction of σ_α:

$$\sigma_\alpha F_\alpha - \sigma_1 F \cos \alpha = 0.$$

By using relation (2.24) we obtain

$$\sigma_\alpha = \sigma_1 \cos^2 \alpha. \tag{2.25}$$

Projecting the forces on the direction of τ_α we have

$$\tau_\alpha F_\alpha - \sigma_1 F \sin \alpha = 0,$$

whence

$$\tau_\alpha = \frac{\sigma_1}{2} \sin 2\alpha. \tag{2.26}$$

For a positive value of σ_1 (i.e., tensile σ_1) and $0 \leqslant \alpha \leqslant 90°$ we obtain a positive value for τ_α. This means that the shearing stress is directed as shown in Fig. 2.35*b*.

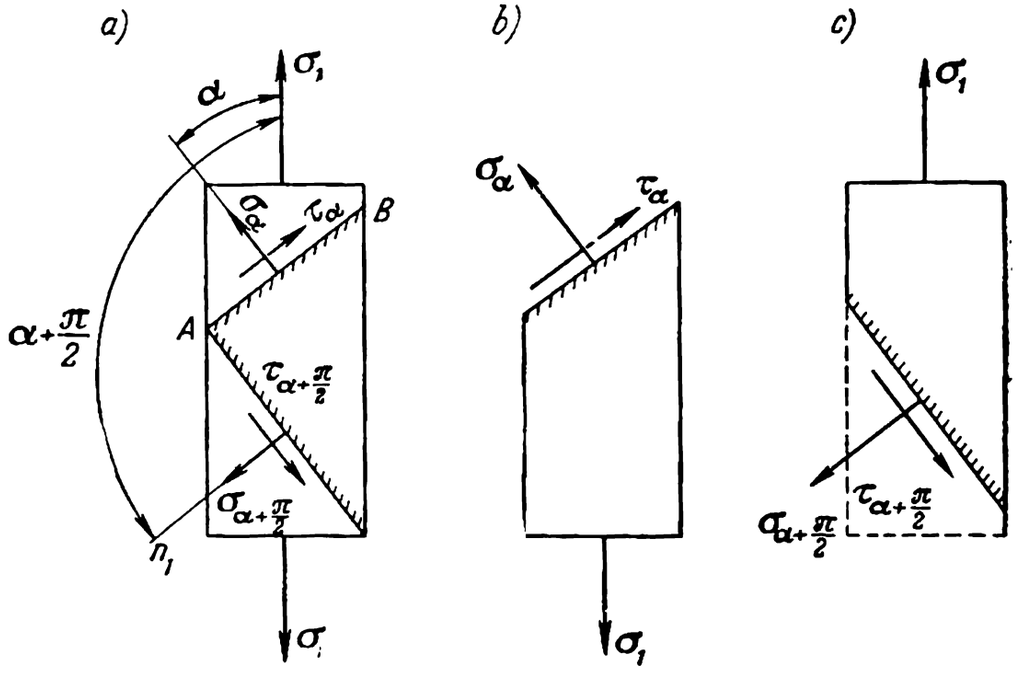


Fig. 2.35

This direction of the shearing stress is characterised by the fact that the normal *n* to the area must be rotated clockwise to coincide with the shearing stress.

Shearing stresses of such direction are commonly considered as positive *.

If, however, the normal to the area must be rotated counter-clockwise to coincide with the shearing stress, then the shearing stress is considered as negative (Fig. 2.35*c*).

From formula (2.25) it follows that normal stresses at-

* In the theory of elasticity another rule is adopted for the signs of shearing stresses.

tain maximum value at $\alpha=0$, i.e., at the section perpendicular to the axis of the bar.

It follows from formula (2.26) that at $\alpha=0$, $\tau=0$, hence the normal stress at the cross section of the bar is a principal stress (see Sec. 7).

It can be seen from formulas (2.25) and (2.26) that at $\alpha=90°$ $\sigma=0$ and $\tau=0$.

Thus there are neither normal nor shearing stresses at longitudinal sections. From formula (2.26) it can also be seen that maximum shearing stresses occur on a section inclined at an angle $\alpha=45°$ and that they are equal to one half of the principal stresses:

$$\tau_{max}=\frac{\sigma_1}{2}.$$

17. Law of Conjugate Shearing Stresses

Let us determine normal and shearing stresses on two areas at right angles to each other.

For an area inclined at an angle α according to formulas (2.25) and (2.26) we have

$$\sigma_\alpha=\sigma_1\cos^2\alpha,$$

$$\tau_\alpha=\frac{\sigma_1}{2}\sin 2\alpha.$$

For an area inclined at an angle $\alpha+\frac{\pi}{2}$ (Fig. 2.35*c*) the normal and shearing stresses may be determined either directly from the condition of equilibrium of the upper or lower portion of the bar (Fig. 2.35*c*) or by formulas (2.25) and (2.26) with α replaced by $\alpha+\frac{\pi}{2}$.

Applying formulas (2.25) and (2.26), we obtain

$$\sigma_{\alpha+\frac{\pi}{2}}=\sigma_1\cos^2\left(\alpha+\frac{\pi}{2}\right)=\sigma_1\sin^2\alpha, \qquad (2.27)$$

$$\tau_{\alpha+\frac{\pi}{2}}=\frac{\sigma_1}{2}\sin 2\left(\alpha+\frac{\pi}{2}\right)=-\frac{\sigma_1}{2}\sin 2\alpha. \qquad (2.28)$$

Examining the results so derived, we see that, first,

$$\sigma_\alpha+\sigma_{\alpha+\frac{\pi}{2}}=\sigma_1,$$

i.e., *the sum of the normal stresses on two areas at right angles to each other is constant and equal to the principal stress*; secondly,

$$\tau_\alpha = -\tau_{\alpha+\frac{\pi}{2}}, \tag{2.29}$$

i.e., *the shearing stresses acting on two areas at right angles to each other are equal in magnitude and opposite in sign (the law of conjugate shearing stresses)*. The shearing stresses on two areas at right angles to each other are both directed either to the intersection of the areas or away from it, as in Fig. 2.35*a*.

If, for example, the sign of σ_1 is changed, the stresses τ_α and $\tau_{\alpha+\frac{\pi}{2}}$ will be reversed and both directed to the intersection A of the areas.

The law of conjugate (reciprocal) shearing stresses is valid not only for a uniaxial state of stress but also for any other (biaxial and three-dimensional) state of stress.

18. Determination of Stresses on Inclined Sections under Tension (Compression) in Two Directions

Let us consider a state of plane stress (Fig. 2.36*a*) when both principal stresses (σ_1 and σ_2) are positive, i.e., tensile.

As has been pointed out previously, the subscripts of the symbols of the principal stresses are chosen so as to satisfy the inequality $\sigma_1 > \sigma_2$. A positive angle α between the σ_1 direction and the normal to an arbitrary area will be taken in a counter-clockwise direction.

The angle between the direction of the stress σ_2 and the area is $\alpha + \frac{\pi}{2}$.

The stresses σ_α and τ_α on an arbitrary oblique section may be either determined from the conditions of equilibrium of the trihedral prism ABC or calculated according to formulas (2.25) and (2.26) by summing up the stresses due to σ_1 and the stresses due to σ_2 (with the angle α replaced by $\alpha + \frac{\pi}{2}$).

We obtain then

$$\sigma_\alpha = \sigma_1 \cos^2 \alpha + \sigma_2 \cos^2 \left(\alpha + \frac{\pi}{2}\right),$$

whence

$$\sigma_\alpha = \sigma_1 \cos^2 \alpha + \sigma_2 \sin^2 \alpha. \tag{2.30}$$

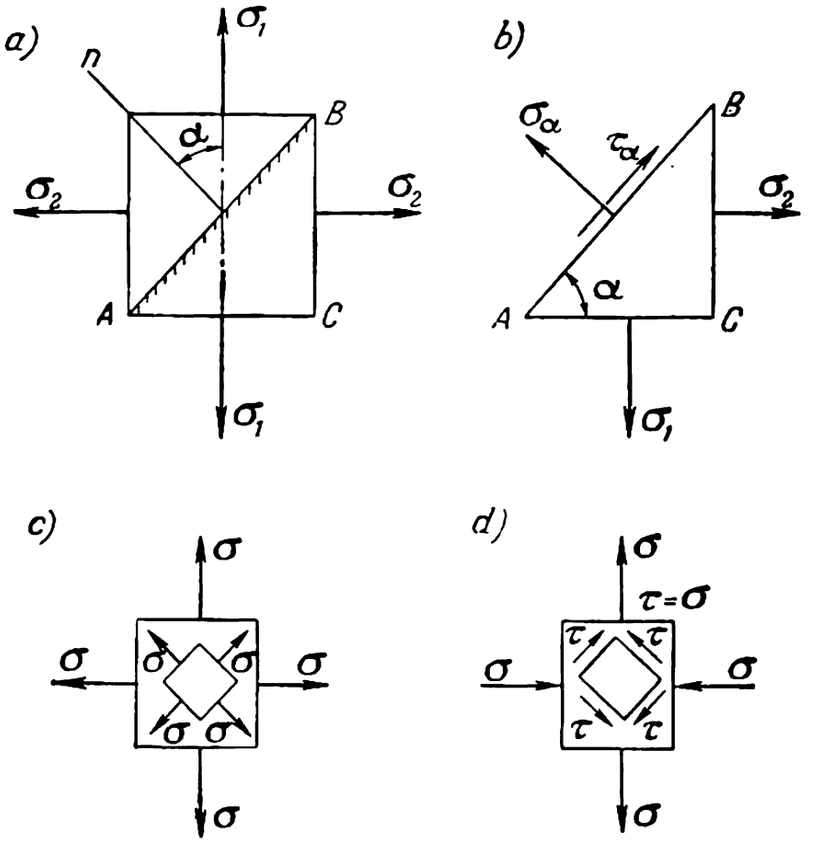


Fig. 2.36

Further

$$\tau_\alpha = \frac{\sigma_1}{2} \sin 2\alpha + \frac{\sigma_2}{2} \sin 2\left(\alpha + \frac{\pi}{2}\right),$$

whence

$$\tau_\alpha = \frac{\sigma_1 - \sigma_2}{2} \sin 2\alpha. \tag{2.31}$$

From formula (2.31) it is seen that the maximum shearing stresses are equal to one-half the difference between the principal stresses

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} \tag{2.32}$$

and they occur on sections equally inclined to the directions σ_1 and σ_2, i.e., at $\alpha = 45°$. This follows from the condition that τ_{max} corresponds to $\sin 2\alpha = 1$.

Having determined the shearing stress on an area perpendicular to the area AB, we see that the law of conjugate shearing stresses holds for a biaxial state of stress. This can also be proved by calculating the values τ_α and $\tau_{\alpha+\frac{\pi}{2}}$ according to formula (2.31).

SPECIAL CASES

Case 1. Consider a state of stress for which $\sigma_1=\sigma_2=\sigma$ (Fig. 2.36*c*).

In this case the shearing stress τ_α is equal to zero and the normal stress has the same value $\sigma_\alpha=\sigma$ on all areas through a given point [see formulas (2.30) and (2.31)]. Such a state of stress is called uniform biaxial tension (or compression).

Case 2. Consider the state of stress represented in Fig. 2.36*d* characterised by the principal stresses $\sigma_1=\sigma$ and $\sigma_2=-\sigma$.

Determine the stresses on sections equally inclined to the directions σ_1 and σ_2, i.e., at $\alpha=45°$ and $\alpha=135°$. According to formulas (2.30) and (2.31) we have: $\sigma_\alpha=0$, $\tau_\alpha=\pm\sigma$. Such a state of stress is called *pure shear.*

19. Determination of Principal Stresses and Position of Principal Areas

Consider the inverse problem. Normal and shearing stresses acting on the faces of an element are given (Fig. 2.37*a*). It is required to determine the position of the principal areas and the magnitude of the principal stresses. Consider the equilibrium of a trihedral prism having a base ABC (Fig. 2.37*b*). Assume that $\sigma_\alpha>\sigma_\beta$. An angle ψ will be measured from the direction of the larger stress towards the normal to the area. A positive direction of the angle ψ will be taken in a counter-clockwise direction. Let the area of the inclined face be denoted by dF. Then the area of the vertical face will be $dF\sin\psi$, and that of the horizontal face $dF\cos\psi$. Projecting all forces on the direction of σ_ψ we obtain

$$\sigma_\psi\,dF-(\sigma_\alpha\,dF\cos\psi)\cos\psi+(\tau\,dF\cos\psi)\sin\psi+$$
$$+(\tau\,dF\sin\psi)\cos\psi-(\sigma_\beta\,dF\sin\psi)\sin\psi=0.$$

Now project all forces on the direction of τ_ψ:

$$\tau_\psi dF - (\sigma_\alpha dF \cos\psi)\sin\psi - (\tau dF \cos\psi)\cos\psi + \\ + (\tau dF \sin\psi)\sin\psi + (\sigma_\beta dF \sin\psi)\cos\psi = 0.$$

Cancelling out dF and introducing functions of double angles, we have

$$\sigma_\psi = \sigma_\alpha \cos^2\psi + \sigma_\beta \sin^2\psi - \tau \sin 2\psi, \tag{2.33}$$

$$\tau_\psi = \frac{\sigma_\alpha - \sigma_\beta}{2} \sin 2\psi + \tau \cos 2\psi. \tag{2.34}$$

With a change in the angle of inclination of the area ψ, the magnitude of σ_ψ and τ_ψ will vary continuously.

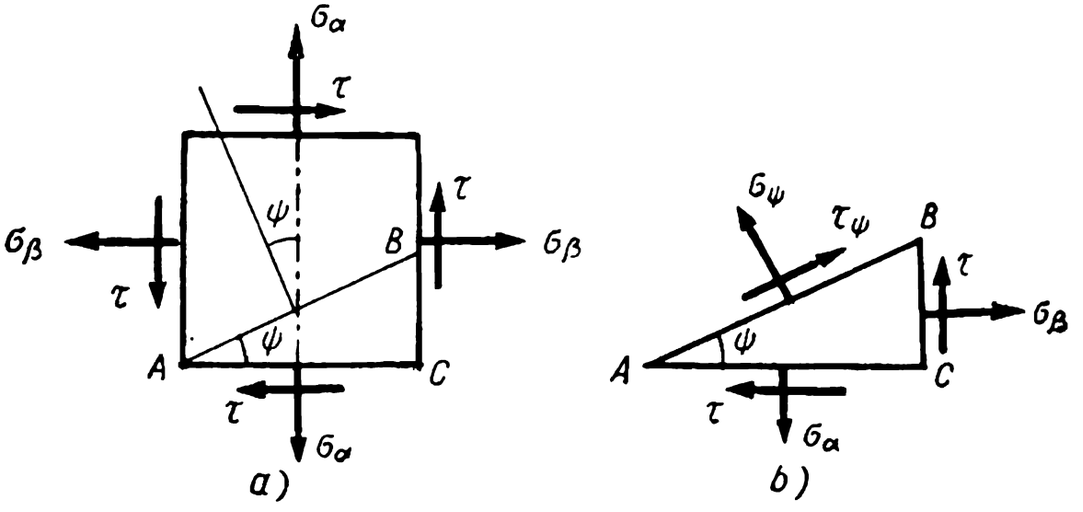


Fig. 2.37

The position of the principal areas, i.e., areas on which extremal normal stresses are acting, can be found either by setting the derivative $\frac{d\sigma_\psi}{d\psi}$ equal to zero or by setting the shearing stresses τ_ψ equal to zero as there are no shearing stresses on the principal areas.

In both cases we obtain the following relation for determining *the angle of inclination* ψ_0 *of the principal areas:*

$$\frac{\sigma_\alpha - \sigma_\beta}{2} \sin 2\psi_0 + \tau \cos 2\psi_0 = 0$$

or

$$\tan 2\psi_0 = \frac{2\tau}{\sigma_\beta - \sigma_\alpha}. \tag{2.35}$$

To obtain extremal values of normal stresses, i.e., magnitudes of the principal stresses, substitute the value of the angle defined by formula (2.35) in formula (2.33). In the first instance trigonometric functions in formula (2.33) should be expressed in terms of the tangent of a double angle. For that use is made of the known formulas of trigonometry:

$$\left.\begin{aligned} \sin 2\psi_0 &= \pm \frac{\tan 2\psi_0}{\sqrt{1+\tan^2 2\psi_0}}, \\ \cos 2\psi_0 &= \pm \frac{1}{\sqrt{1+\tan^2 2\psi_0}}, \\ \cos^2 \psi_0 &= \frac{1}{2}(1+\cos 2\psi_0), \\ \sin^2 \psi_0 &= \frac{1}{2}(1-\cos 2\psi_0). \end{aligned}\right\}$$

After simple transformations we obtain the following *formula for determining magnitudes of the principal stresses*:

$$\sigma_{\substack{\max \\ \min}} = \frac{\sigma_\alpha + \sigma_\beta}{2} \pm \frac{1}{2}\sqrt{(\sigma_\alpha - \sigma_\beta)^2 + 4\tau^2}. \qquad (2.36)$$

If one of the given normal stresses is equal to zero, formula (2.36) will be simplified and will become

$$\sigma_{\substack{\max \\ \min}} = \frac{\sigma}{2} \pm \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2}. \qquad (2.37)$$

We shall use this formula subsequently when studying bending and combined loading. Analysing the second derivative $\frac{d^2\sigma_\psi}{d\psi^2}$ one can see that the maximum principal stress acts on the principal area at an angle ψ_0 under the assumptions made ($\sigma_\alpha > \sigma_\beta$), and the minimum principal stress acts on the area at an angle $\psi_0 + 90°$.

Example 2.10. Determine the magnitude and direction of the principal stresses when the state of stress is as shown in Fig. 2.38.

Solution. According to formula (2.35) we have

$$\tan 2\psi_0 = \frac{2 \times 100}{200 - 300} = -2; \quad 2\psi_0 = -63°30'; \quad \psi_0 = -31°45'.$$

The negative sign indicates that ψ_0 is measured from the direction of $\sigma_\alpha = 300$ kg/cm² in a clockwise direction.

According to formula (2.36) we obtain

$$\sigma_{max} = \frac{300+200}{2} + \frac{1}{2}\sqrt{(300-200)^2+4\times 100^2} =$$

$$= 250 + 112 = 362 \text{ kg/cm}^2,$$

$$\sigma_{min} = 250 - 112 = 138 \text{ kg/cm}^2.$$

In accordance with the foregoing σ_{max} acts on the area at an angle ψ_0, σ_{min} acts on the area at an angle $\psi_0+90°$.

Example 2.11. What will be changed in the preceding example if the shearing stresses act in the opposite direction?

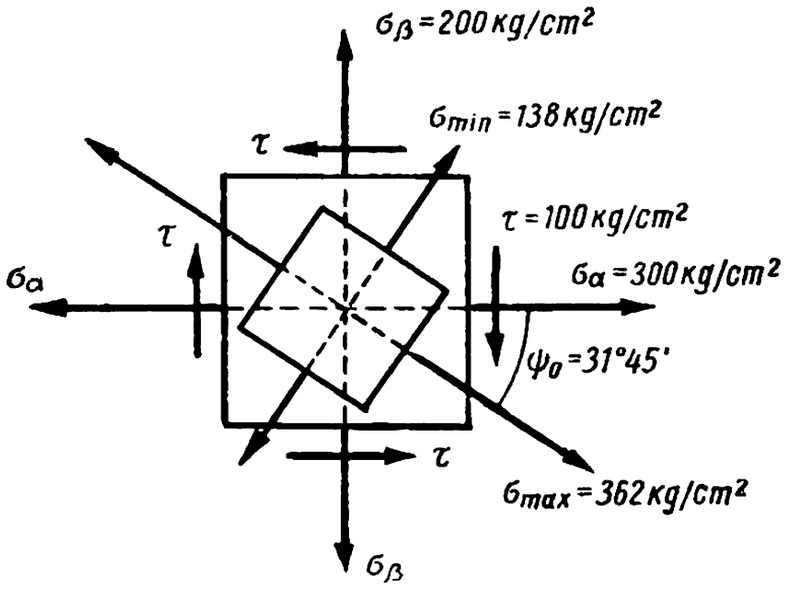


Fig. 2.38

Answer. Since the shearing stress τ will become negative, there will be a change in the sign of the angle of inclination of the principal areas, which will be $\psi_0 = 31°45'$ (measured in a counter-clockwise direction). The magnitude of the principal stresses will not change.

20. Relation Between Strains and Stresses in Plane and Three-Dimensional States of Stress (Generalised Hooke's Law)

Let us determine strains ε_1 and ε_2 in the direction of the principal stresses under plane stress conditions (Fig. 2.39). For that we shall use Hooke's law for the uniaxial state of stress (see formula 2.3) and also relation (2.5) between longitudinal and lateral deformations and

the principle of superposition (the principle of adding up deformations).

Due to stress σ_1 alone the unit elongation in the vertical direction is

$$\varepsilon_{11} = \frac{\sigma_1}{E}$$

and at the same time the unit contraction in the horizontal direction is

$$\varepsilon_{21} = -\mu \frac{\sigma_1}{E}.$$

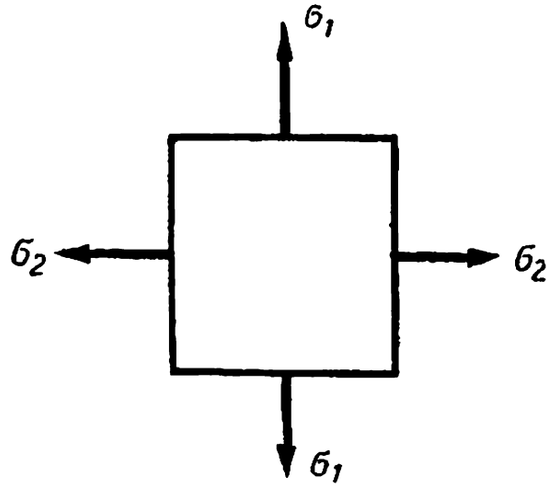


Fig. 2.39

Due to σ_2 alone we would have: in the horizontal direction the elongation

$$\varepsilon_{22} = \frac{\sigma_2}{E},$$

and in the vertical direction the contraction

$$\varepsilon_{12} = -\mu \frac{\sigma_2}{E}.$$

By adding together the deformations we obtain equations

$$\left.\begin{aligned} \varepsilon_1 &= \varepsilon_{11} + \varepsilon_{12} = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}, \\ \varepsilon_2 &= \varepsilon_{22} + \varepsilon_{21} = \frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E}, \end{aligned}\right\} \qquad (2.38)$$

representing *generalised Hooke's law* for plane stress.

If strains ε_1 and ε_2 are known, then in solving equations (2.38) with respect to stresses σ_1 and σ_2 we obtain the following formulas:

$$\left.\begin{aligned} \sigma_1 &= \frac{E}{1-\mu^2}(\varepsilon_1 + \mu\varepsilon_2), \\ \sigma_2 &= \frac{E}{1-\mu^2}(\varepsilon_2 + \mu\varepsilon_1). \end{aligned}\right\} \qquad (2.39)$$

Similarly, for a three-dimensional state of stress when all three principal stresses σ_1, σ_2 and σ_3 are other than

zero, we obtain formulas:

$$\left.\begin{aligned}\varepsilon_1 &= \frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)],\\ \varepsilon_2 &= \frac{1}{E}[\sigma_2 - \mu(\sigma_3 + \sigma_1)],\\ \varepsilon_3 &= \frac{1}{E}[\sigma_3 - \mu(\sigma_1 + \sigma_2)],\end{aligned}\right\} \tag{2.40}$$

representing generalised Hooke's law for three-dimensional stress. The strains ε_1, ε_2 and ε_3 in the direction of the principal stresses are called *the principal strains.*

Knowing ε_1, ε_2 and ε_3 we can calculate *the change in volume* due to deformation. Let us take a cube $1 \times 1 \times 1$ cm. Its volume before deformation is $V_0 = 1$ cm^3. The volume after deformation is

$$V_1 = (1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3) \approx 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

(the products of ε are ignored as being small quantities compared to ε itself).

The unit volume change ε_V is

$$\varepsilon_V = \frac{V_1 - V_0}{V_0} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3. \tag{2.41}$$

Substituting here the values of ε_1, ε_2 and ε_3 given by (2.40) we obtain

$$\varepsilon_V = \frac{1 - 2\mu}{E}(\sigma_1 + \sigma_2 + \sigma_3). \tag{2.42}$$

From formula (2.42) it follows that Poisson's ratio cannot be greater than 0.5. Indeed, it is obvious that the volume of an element cannot diminish in triaxial tension, i.e., ε_V is positive, which is possible only if $1 - 2\mu \geqslant 0$, as the principal stresses are positive in this case ($\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 > 0$).

21. Work of External and Internal Forces in Tension (Compression). Potential Energy of Strain

In tension (compression) external forces do work by reason of the displacement of their points of application (Fig. 2.40*a*).

Let us calculate the work done by a statically applied external force, i.e., a force whose magnitude increases very slowly during deformation from zero to its final value.

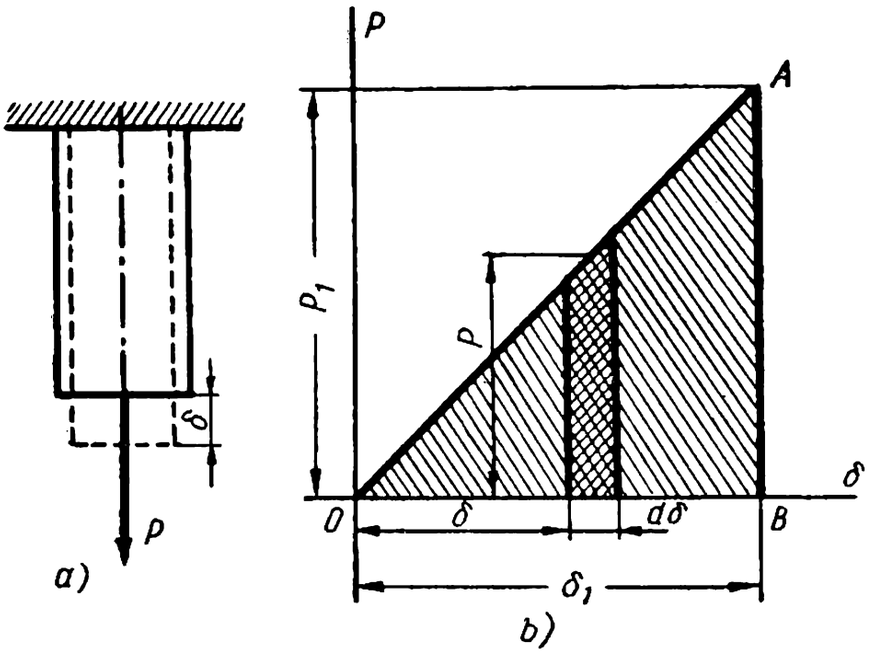


Fig. 2.40

The elementary work dA done by the external force P on the displacement $d\delta$ is

$$dA = P\,d\delta. \tag{2.43}$$

But there exists a relation between δ and P (Hooke's law)

$$\delta = \frac{Pl}{EF},$$

whence

$$P = \frac{EF\delta}{l}.$$

Substituting this value in formula (2.43) we have

$$dA = \frac{EF}{l}\,\delta\,d\delta.$$

The total work done by the force will be obtained by integrating this expression from zero to the final value of displacement δ_1

$$A = \frac{EF}{l}\int_0^{\delta_1} \delta\,d\delta = \frac{EF\,\delta_1^2}{2l} = \frac{P_1\delta_1}{2}.$$

Thus

$$A = \frac{1}{2} P_1 \delta_1, \tag{2.44}$$

i.e., *the work done by an external statically applied force is equal to one-half the product of the final magnitude of the force and the final magnitude of the corresponding displacement.*

The work done by force P is represented graphically (with regard to the scales) by the area OAB under the curve, plotted in the co-ordinate system δ-P. During deformation not only external forces do work but also internal forces (elastic forces).

The work done by internal forces in tension (compression) can be calculated as follows.

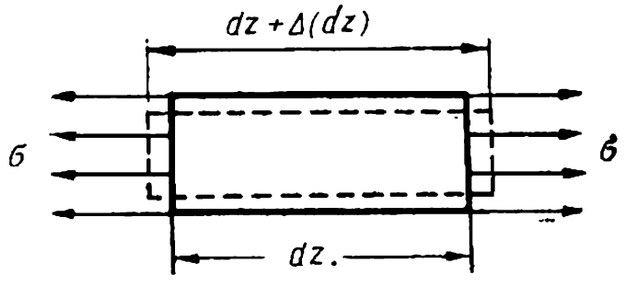


Fig. 2.41

Fig. 2.41 shows an element dz of a bar acted upon by normal stresses σ which are external forces for this element.

Internal forces will obviously act in the opposite direction, i.e., in the direction opposite to the displacement. Therefore *the work done by internal forces under loading is always negative.*

The elementary work of the internal forces (for the element dz) is calculated according to a formula analogous to (2.44):

$$dU = -\frac{1}{2} N \Delta (dz), \tag{2.45}$$

where $\Delta(dz)$ is the elongation of the element, N is the internal force (the longitudinal force). But by virtue of Hooke's law we have

$$\Delta (dz) = \frac{N\,dz}{EF}.$$

Consequently,

$$dU = -\frac{1}{2} \frac{N^2\,dz}{EF}. \tag{2.46}$$

The total work done by the internal forces will be obtained by integrating both parts of formula (2.46) over the length of the entire bar l:

$$U = -\frac{1}{2}\int_0^l \frac{N^2\,dz}{EF}. \tag{2.47}$$

If N, E and F are constant, then

$$U = -\frac{1}{2}\frac{N^2 l}{EF} = -\frac{EF\,\Delta l^2}{2l}, \tag{2.48}$$

where $\Delta l = \delta = \frac{Nl}{EF}$ is the elongation of the bar.

The quantity equal to the work done by internal forces but having an opposite sign is called *the potential energy of strain.* It represents the energy stored by a body during deformation.

Thus, for a bar of constant section subjected to a longitudinal force having the same value at all cross sections the potential energy in tension (compression) is determined according to the formula

$$\Pi = -U = \frac{1}{2}\frac{N^2 l}{EF} = \frac{EF\,\Delta l^2}{2l}. \tag{2.49}$$

The potential energy per unit volume of the material is called *the unit potential energy*:

$$u = \frac{\Pi}{V} = \frac{\Pi}{Fl} = \frac{N^2 l}{2EF^2 l} = \frac{\sigma^2}{2E}, \tag{2.50}$$

or

$$u = \frac{1}{2}E\varepsilon^2 \quad (\text{since } \sigma = E\varepsilon),$$

or

$$u = \frac{1}{2}\sigma\varepsilon. \tag{2.51}$$

In a three-dimensional state of stress the unit potential energy will be obtained as the sum of three terms:

$$u = \frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3). \tag{2.52}$$

By using generalised Hooke's law we have

$$u = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]. \tag{2.53}$$

Assuming one of the principal stresses to be equal to zero the formula for a state of plane stress can readily be obtained from this formula as a particular case.

Experiments show that for rates of loading most commonly encountered in practice the work of external forces is almost completely consumed in the storage of potential energy of strain. Slight losses of energy, mainly in the heating of a part, are of no practical importance and can be neglected.

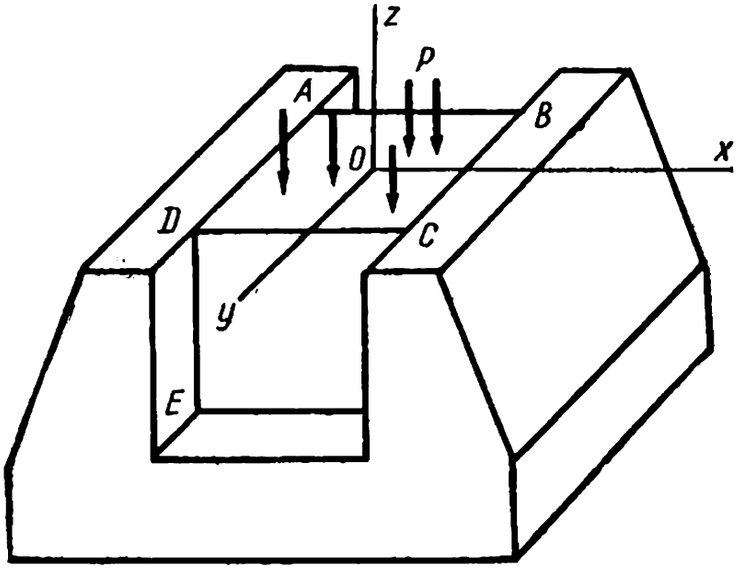


Fig. 2.42

Example 2.12. Determine stresses σ_1 and σ_2 if strains in these directions are $\varepsilon_1=0.001$, $\varepsilon_2=-0.0008$, the modulus of elasticity $E=2\times10^6$ kg/cm^2, Poisson's ratio $\mu=0{,}3$.

Solution. According to formula (2.39) we have

$$\sigma_1=\frac{E}{1-\mu^2}(\varepsilon_1+\mu\varepsilon_2)=$$

$$=\frac{2\times10^6}{1-0.3^2}(0.001-0.3\times0.0008)=1{,}670\ \text{kg/cm}^2,$$

$$\sigma_2=\frac{E}{1-\mu^2}(\varepsilon_2+\mu\varepsilon_1)=$$

$$=\frac{2\times10^6}{1-0.3^2}(-0.0008+0.3\times0.001)=-1{,}100\ \text{kg/cm}^2.$$

Example 2.13. A rubber cube $ABCD$ is inserted loosely but without gaps into a steel form so that its two opposite faces are free (Fig. 2.42). The cube is subjected to pressure p kg/cm^2 on the upper face, Determine the stress

σ_x, and the strains ε_y and ε_z, as well as the unit volume change. The modulus of elasticity of rubber is E, Poisson's ratio is μ. Friction between the cube and the walls should be neglected. The steel form should be assumed to be perfectly rigid (undeformable).

Solution. By virtue of generalized Hooke's law we have

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \mu(\sigma_y + \sigma_z)],$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \mu(\sigma_z + \sigma_x)],$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \mu(\sigma_x + \sigma_y)].$$

According to the conditions of the problem $\sigma_y = 0$; $\sigma_z = -p$; $\varepsilon_x = 0$. By using these conditions we obtain

$$\sigma_x = -\mu p; \quad \varepsilon_y = \frac{\mu(\mu+1)}{E} p; \quad \varepsilon_z = -\frac{1+\mu^2}{E} p;$$

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z = -\frac{(1-2\mu)(1+\mu)}{E} p.$$

22. Stress Concentration. Contact Stresses

A uniform distribution of stresses over the cross section of a bar under tension (compression) takes place only at some distance from the point of application of a force and provided the lateral dimensions of the bar do not change at all or change very gradually along its length. If the contour of a longitudinal section of the bar does change abruptly, then at the places where there is an abrupt discontinuity in the prismatical or cylindrical form of the bar stress distribution over its cross section will no longer be uniform.

This phenomenon of an abrupt increase in stresses at places of sharp changes in the geometrical form of a bar is called *stress concentration.* The determination of stresses at places of concentration is carried out experimentally or by methods of the theory of elasticity.

As an example Fig. 2.43*a* presents the results obtained in solving this problem for a wide strip with a small hole when subjected to uniform tension. The maximum stress at the edge of the hole σ_{max} is three times the average

(nominal) stress as calculated according to the formula $\sigma = \frac{P}{F}$.

For a wide strip having small semicircular grooves at the edges the maximum stress is twice as large as the nominal stress (Fig. 2.43*b*).

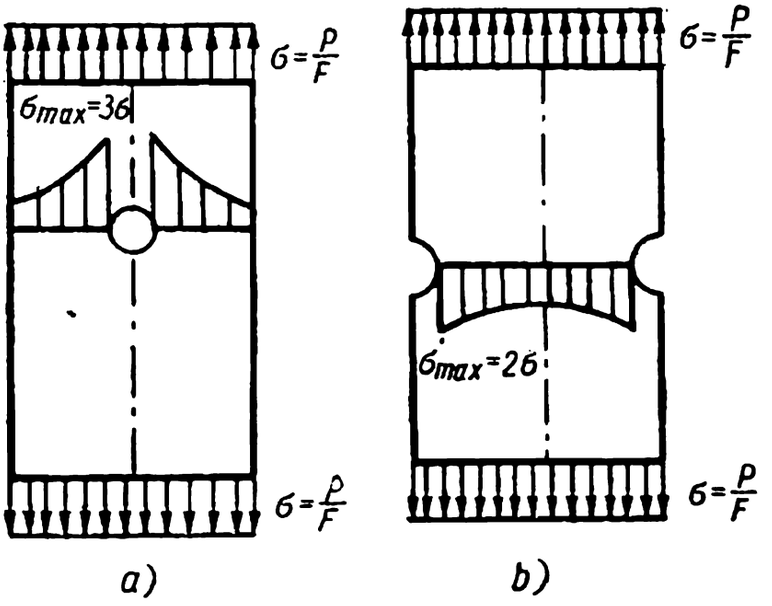


Fig. 2.43

The ratio of maximum local stress to nominal stress is called *the theoretical stress concentration factor*:

$$\alpha_\sigma = \frac{\sigma_{max}}{\sigma}. \qquad (2.54)$$

The nominal stress is that which is determined by formulas of strength of materials with no consideration of the stress concentration effect. For example, for a strip with a hole

$$\sigma = \frac{P}{F},$$

where F is the area of the weakened section.

The theoretical stress concentration factor determined on the assumption that the material follows Hooke's law during deformation in many cases does not give a correct idea of the effect of stress concentration on the strength of a part. If the material followed Hooke's law up to fracture, the strength of a part in the presence of stress concentration would be less than the strength of a similar part without stress raisers by a factor of α_σ (theoretical

stress concentration factor). Experiments show that for most materials the reduction in strength due to stress concentration is less than by a factor of α_σ. This reduction is determined experimentally as the ratio of the ultimate strength (σ_u) of a part without stress concentration to the ultimate strength (σ_{u_c}) of a part having a given stress raiser

$$k_s = \frac{\sigma_u}{\sigma_{u_c}}. \tag{2.55}$$

Factor k_s is called *the effective stress concentration factor*. Values of k_s are given in handbooks.

Experiments show that under static loadings for parts made of ductile materials k_s is in practice equal to unity, i.e., under such loadings stress concentration has to be taken into consideration only in the design of parts made of brittle or low-ductile materials.

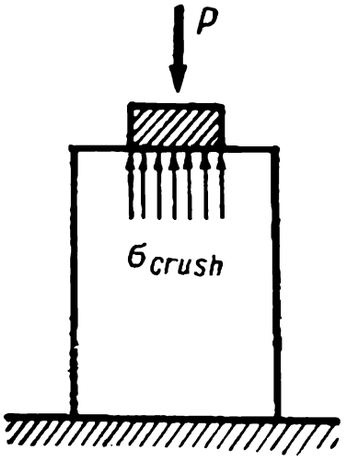


Fig. 2.44

Under repeated loading (fatigue analysis) stress concentration is taken into consideration for all materials.

High local stresses also occur at places where pressure is transmitted from one body to another; they are called contact stresses, or crushing stresses. Their magnitude decreases very rapidly with an increase in the distance from the area of contact between the parts.

If the bodies had flat surfaces before coming into contact (Fig. 2.44), the distribution of stresses over the contact area may be assumed to be uniform and the stresses may be determined by using the formula

$$\sigma_{crush} = \frac{P}{F}. \tag{2.56}$$

If the bodies had curved surfaces before coming into contact, as in the case of bearings, for instance, the problem of determining contact stresses is complicated and can be solved only by the methods of the theory of elasticity. The problem is solved on the basis of the following assumptions:

(a) the materials of the parts in contact obey Hooke's law;

(b) the linear dimensions of the contact area are small as compared to the radii of curvature of the contacting surfaces;

(c) the compressive force is directed along the normal to the contact area;

(d) only normal stresses occur at the contact surface.

Under these assumptions, as is proved in the theory of elasticity, the normal stresses are distributed over the contact area according to an ellipsoid surface law and the contact area has in general the form of an ellipse. The maxi-

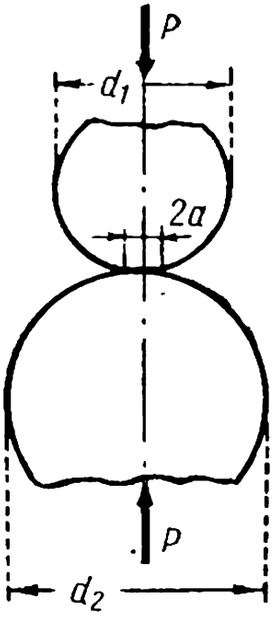


Fig. 2.45

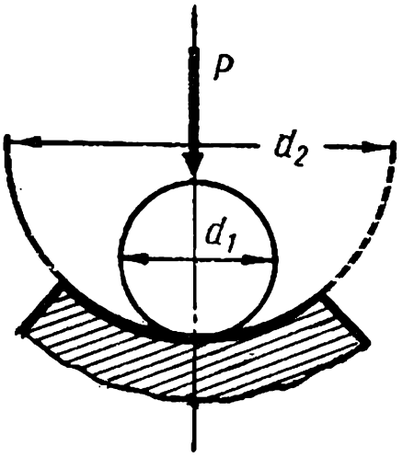


Fig. 2.46

mum stress occurs at the centre of the contact area. Let us consider some design formulas for special cases of contact deformation.

1. When two elastic spheres having diameters d_1 and d_2 (Fig. 2.45) are pressed against each other, a circular contact area results, the radius a of which can be determined according to the formula

$$a = 0.88 \sqrt[3]{\frac{P d_1 d_2}{E(d_1 + d_2)}}. \qquad (2.57)$$

The maximum crushing stress at the centre of the area is given by

$$\sigma_{max} = 0.62 \sqrt[3]{P E^2 \left(\frac{d_1 + d_2}{d_1 d_2}\right)^2}. \qquad (2.58)$$

In deriving these formulas Poisson's ratio is taken as $\mu = 0.3$. Formula (2.58) is also valid in the case of a sphere d_1 in diameter resting against a concave spherical surface d_2 in diameter (Fig. 2.46); only in this case the

sign of d_2 should be changed to its opposite. The result is

$$\sigma_{max} = 0.62 \sqrt[3]{PE^2 \left(\frac{d_2 - d_1}{d_1 d_2}\right)^2}. \qquad (2.59)$$

The stresses are less than in the preceding case.

In the case of a sphere resting on a plane the stresses are determined from formula (2.59) if d_2 is assumed to be infinite:

$$\sigma_{max} = 0.62 \sqrt[3]{\frac{PE^2}{d^2}}. \qquad (2.60)$$

For other cases formulas for determining contact stresses are given in handbooks or in complete courses on strength of materials.

Experience shows that materials are capable of sustaining very high contact stresses. This is due to the fact that these stresses decrease rapidly as the distance from the contact zone increases and also to the fact that the material near the contact area undergoes triaxial compression rather than uniaxial. As will be shown later (see Chapter VIII), the material is able to sustain considerable stresses under triaxial compression.

Table 2.8 gives values of allowable contact stresses $[\sigma_{cont}]$ for a number of materials.

If the dimensions of the contact area are comparable to the magnitude of the radii of curvature of the surfaces in contact, the foregoing formulas are not applicable.

Such a problem is encountered, for instance, in determining pressure between the surface of the body of a bolt (or a rivet) and the cylindrical surface of a hole. In these cases the theoretical solution is very complicated and for checking the strength of a material in the zone of the contact area use is usually made of engineering methods of analysis based on experiments. For instance, in the design of rivets and bolts required to withstand crushing it is assumed that stresses are distributed uniformly over the crushing area and the allowable crushing stresses are assumed on the basis of experimental data to be 2-2.5 times as large as the allowable tensile (compressive) stresses

$$[\sigma_{crush}] = 2\text{-}2.5\,[\sigma].$$

Table 2.8

Material	Ultimate strength σ_u in kg/mm²	$[\sigma_{cont}]$ in kg/cm²
	Steel	
30	48-60	8,500-10,500
40	57-70	10,000-13,500
50	63-80	10,500-14,000
50Г	65-85	11,000-14,500
20Х	70-85	12,000-14,500
	Cast iron	
СЧ21-40	95	8,000-9,000
СЧ22-44	100	9,000-10,000

In the presence of shearing stresses at the contact area (gear and screw drives, etc.) as well as under repeated loads contact stresses and allowable stresses are determined according to formulas given in courses on machine elements.

CHAPTER III

SHEAR

23. Determination of Stresses

When a transverse force Q alone is acting on cross sections of a bar and the other internal forces are equal to zero, this type of deformation is called *shear*. In this case only shearing stresses arise at the section.

In most practical problems a transverse force acts simultaneously with a bending moment and a longitudinal force so that normal stresses as well as shearing stresses usually act over the cross sections.

However, in cases where shearing stresses are considerably larger than normal stresses, shearing strength analysis only need be done.

A typical example of such a simplified but, as experience shows, quite reliable analysis is calculation of the strength of riveted, bolted and welded joints.

Fig. 3.1*a* shows two sheets joined by means of rivets (lap joint).

The other (closing) rivet head is formed in the process of riveting.

A picture of a possible failure of a riveted joint is shown in Fig. 3.1*b*. The joint fails due to shear of rivets along the line of contact of the sheets.

If failure of each rivet occurs along a single shear plane, the rivet joint is referred to as *single-shear* (Fig. 3.1), if along two planes the joint is referred to as *double-shear* (Fig. 3.6), and so on.

Shearing stresses act over rivet shear planes. To determine the magnitude of these shearing stresses it is necessary to know first of all how the force P is distributed between individual rivets. To do this the rivet joint should be calculated as a statically indeterminate system.

The redundant unknowns may be taken to be the forces between rivets in one of the sheets (Fig. 3.2). (Other variants of the basic system are also possible). Equating to zero the displacements at the places where the sheets are cut, we obtain the equations of deformations necessary for the determination of the unknown forces.

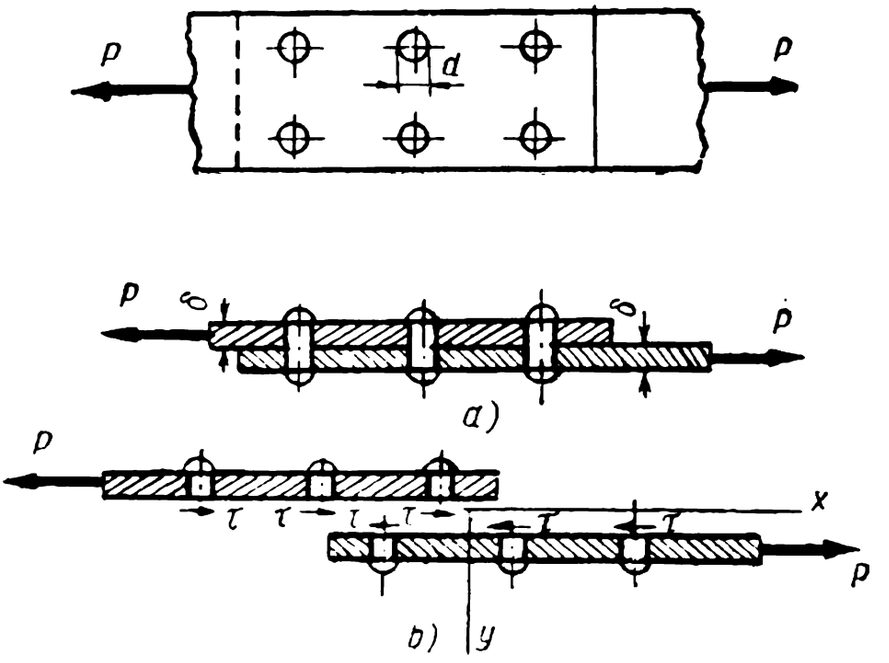


Fig. 3.1

After determining the forces in the sheets we find the shearing forces in the rivets as the difference of the forces adjacent to the rivet; for example, the shearing force in the third rivet (Fig. 3.2) is $Q_3 = X_3 - X_2$, etc.

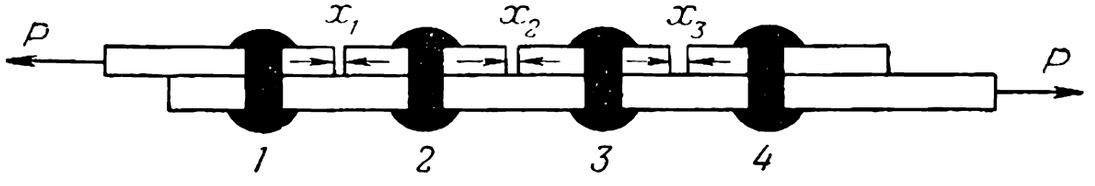


Fig. 3.2

The results of solution of this problem for a lap joint with the same cross-sectional area of the sheets being riveted are presented in Table 3.1.

It can be seen that unequal loading conditions for rivets become more distinct with increase in number of rivets in a row.

Table 3.1

Number of rivets	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_m
	expressed as fractions of P						
3	0.353	0.294	0.353				0.333
4	0.29	0.21	0.21	0.29			0.25
5	0.26	0.17	0.14	0.17	0.26		0.20
6	0.24	0.15	0.11	0.11	0.15	0.24	0.166

With six rivets the shearing forces in the outer rivets (first and sixth) are about 2.5 times larger than in the middle rivets (third and fourth).

In cases where elements of different cross-sectional area are connected unequal loading conditions for rivets are accentuated. The most overstressed rivets are on the side of the sheet having the smaller cross-sectional area.

Experiments show, however, that the rivets fail simultaneously under static loading. This is due to the fact that the forces in the rivets are equalised by the moment of failure due to ductility of the material and due to the clearances between the rivets and the sheets.

When impact and vibratory loads are in question unequal loading conditions for rivets must be taken into consideration.

Thus, under the action of a static load it may be taken that the shearing force in each rivet is

$$Q = \frac{P}{n}, \tag{3.1}$$

where P is the force acting on the joint, n is the number of rivets.

In the case of a rivet joint in double shear (see, for instance, Fig. 3.6) n denotes the number of rivets located on one side of the joint of the sheets being connected.

In addition, the assumption is made that shearing stresses are distributed uniformly over the shear plane though actually, as experimental investigations show, their distribution is not uniform. However, a strict theoretical solution of this problem is rather involved, the more so as there are clearances between rivets and sheets, fric-

tional forces between sheets, etc. Furthermore, rivets are usually made of the most ductile steels, and hence irregularity in the distribution of shearing stresses due to occurrence of plastic deformations disappears by the moment of failure.

Having assumed the shearing stresses to be uniformly distributed over the cross section of the rivet we can easily find their magnitude.

Deriving the equation of equilibrium of the cut portion of the joint, for instance, of the upper portion (Fig. 3.1*b*), we obtain

$$\Sigma X = 0; \quad -P + n\tau F = 0;$$

whence

$$\tau = \frac{P}{nF}, \tag{3.2}$$

where $F = \frac{\pi d^2}{4}$ is the cross-sectional area of a rivet d in diameter.

By using formula (3.1) we have from (3.2)

$$\tau = \frac{Q}{F}. \tag{3.3}$$

The condition of shearing strength of rivets is as

$$\tau = \frac{P}{Fn} = \frac{Q}{F} \leqslant [\tau_{sh}]. \tag{3.4}$$

Here $[\tau_{sh}]$ is the allowable shearing stress.

From formula (3.4) the necessary number of single-shear rivets can readily be determined:

$$n \geqslant \frac{P}{\frac{\pi d^2}{4}[\tau_{sh}]}. \tag{3.5}$$

With double-shear or multishear rivet joints the total number of rivet shear planes located on one side of the joint should be substituted for n in formula (3.4).

The magnitude of allowable shearing stresses is usually established experimentally to reveal the influence of irregularity in stress distribution on the strength of a joint, the influence of frictional forces, of clearances, etc. In the design of rivets it is assumed that $[\tau_{sh}] = (0.6\text{-}0.8)\ [\sigma_t]$ where $[\sigma_t]$ is the allowable tensile stress.

Experiments have shown that fibrous materials (such as wood), which are highly non-homogeneous and anisotropic,

possess very low resistance to shear parallel to the grain. For instance, for pine subjected to shear parallel to the grain it is assumed that $[\tau_{sh}]=0.1\ [\sigma_t]$.

Shearing stresses act (Fig. 3.3) not only over the shear areas *BC* and *AD* but also, as follows from the law of conjugate shearing stresses, over the areas *CD* and *AB* perpendicular to the former.

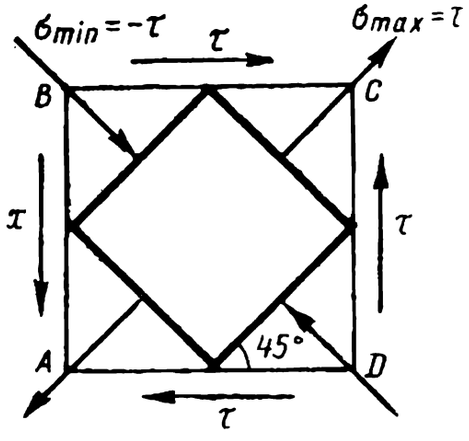


Fig. 3.3

Both normal and shearing stresses act on inclined sections. The maximum normal stresses occur on the principal areas. Their magnitude and direction can be determined from formulas (2.35) and (2.36).

Since in our case $\sigma_\alpha=\sigma_\beta=0$ formula (2.35) gives $\tan 2\psi_0=\infty$. Consequently, the principal areas are inclined at an angle $\psi_0=45°$ to the direction of the shear areas (see Fig. 3.3). The principal stresses defined by formula (2.36) are

$$\sigma_{\substack{\max\\ \min}}=\pm\tau, \tag{3.6}$$

i.e., they are equal in magnitude to the shearing stresses acting on the shear planes; one of the principal stresses will be tensile, the other compressive (see Fig. 3.3). Since two principal stresses are other than zero, shear, as has been pointed out above, is a particular case of the plane (biaxial) state of stress.

In addition to shear, *crushing* also has to be provided against when rivet joints are designed. Crushing stresses are checked over the contact area between the rivets and sheets being connected. The projections of the crushing areas on the plane of the drawing are shown in Fig. 3.1*a* by thick lines. The crushing area of one rivet is assumed to be $F_{crush}=d\times\delta$. * The crushing stresses are considered

* In calculating F_{crush} in the case of riveted lap joints connecting two sheets of different thickness δ_{min} should be taken. For a butt joint with two cover plates (see Fig. 3.6) δ should represent the thickness of the sheets being connected if it does not exceed $2\delta_{pl}$ where δ_{pl} is the thickness of one cover plate; otherwise (at $2\delta_{pl}<\delta$) one should take $F_{crush}=2d\delta_{pl}$.

to be uniformly distributed over the crushing area and the condition of crushing strength is expressed by the formula

$$\sigma_{crush} = \frac{P}{n' F_{crush}} \leqslant [\sigma_{crush}], \tag{3.7}$$

where $[\sigma_{crush}]$ is the allowable crushing stress, n' is the number of rivets. For drilled or punched and then reamed holes it is assumed that $[\sigma_{crush}] = 2[\sigma]$.

From formula (3.7) the necessary number of rivets can be determined on the basis of crushing strength:

$$n'_{crush} \geqslant \frac{P}{d\delta\,[\sigma_{crush}]}. \tag{3.8}$$

Of the two quantities n and n' the greater is taken.

By use of the formula $\tau = \frac{Q}{F}$ welded joints can also be designed—these have been progressively taking the place

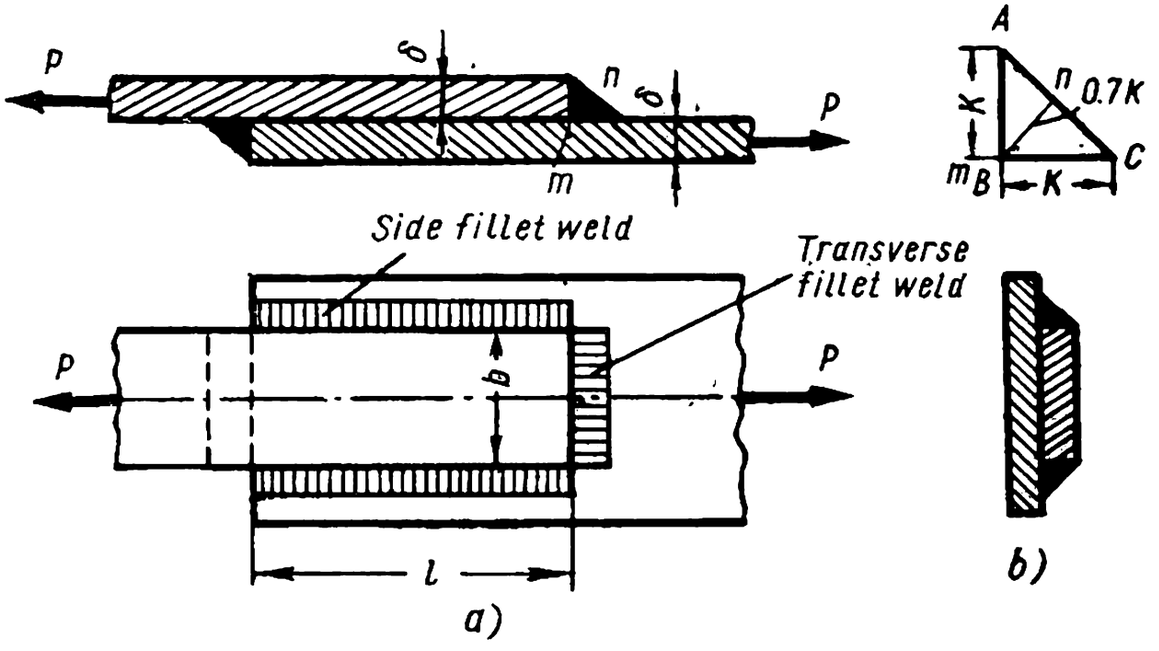


Fig. 3.4

of riveted joints in recent years. Fig. 3.4*a* shows a lap joint of two sheets by means of transverse and side fillet welds. In designing both transverse and side fillet welds it is assumed that the dangerous section in the weld coincides with the plane passing through the bisector *mn* of the right angle *ABC* (Fig. 3.4*b*). Thus, for one transverse fillet weld the area of the dangerous section is $b\ 0.7k$, and for one side fillet weld $l\ 0.7k$ where k is the leg of the weld; in the case represented in Fig. 3.4 the leg of the

weld is equal to the thickness δ of the upper sheet. The shearing stresses are assumed to be uniformly distributed over the area of the dangerous section. Based on the above assumptions the allowable load for a transverse fillet weld is determined by the formula

$$[P_{tr}] = b\,0.7\,k\,[\tau_w] \tag{3.9}$$

where $[\tau_w]$ is the allowable shearing stress for the weld.

The allowable load for one side fillet weld is

$$[P_s] = l\,0.7\,k\,[\tau_w]. \tag{3.10}$$

Strength considerations obviously require that the total allowable resistance of the welds should not be less than the force acting on the joint:

$$2\,[P_s] + 2\,[P_{tr}] \geqslant P. \tag{3.11}$$

By using this equation, it is possible, given the dimension k, to determine the required length of welds.

There are widely differing types of rivet and welded joints. They are treated in detail in the textbook courses on machine elements and in special courses on the subject.

24. Deformations in Shear

The element $ABCD$, rectangular before deformation, (Fig. 3.5) will assume the form $AB'C'D$ after deformation by shearing (the face AD is considered to be fixed).

The quantity $CC' = \Delta S$ is called *the absolute shear.*

The ratio $\frac{\Delta S}{a} = \tan\gamma$ is called *the shearing strain.*

Since the deformations are small it is permissible to assume $\tan\gamma \approx \gamma$. This angle is called *the shear angle.*

Experiments show that for many materials up to certain limits of loading a linear relationship exists between stresses and strains in shear:

$$\gamma = \frac{\tau}{G} \tag{3.12}$$

which represents *Hooke's law in shear.* The constant G is called *the shearing modulus* (the modulus of elasticity of the second kind), it characterises the ability of a material

to resist shearing. If γ is known the absolute shear can be found as:

$$\Delta S = \gamma a = \frac{\tau a}{G} = \frac{Qa}{GF}, \tag{3.13}$$

where Q is the force acting on the face BC, and F is the area of this face (here, as before, shearing stresses are assumed to be distributed uniformly over the area of their action).

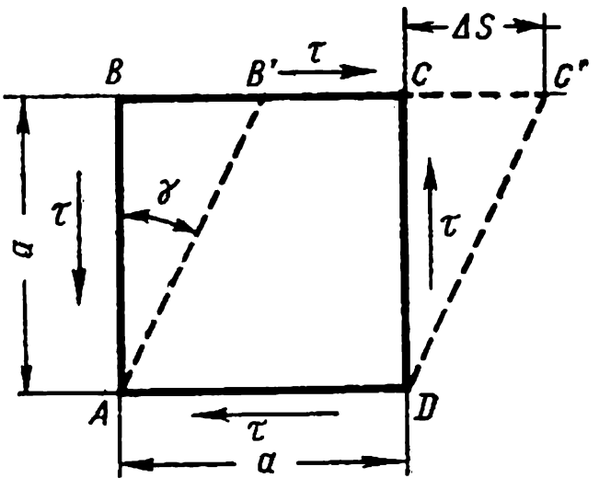


Fig. 3.5

The linear relation between τ and γ is valid so long as shearing stresses do not exceed the proportional limit in shear.

From formula (2.42) it is seen that in pure shear (considering that $\sigma_1 = \tau$ and $\sigma_3 = -\tau$, $\sigma_2 = 0$) the volumetric strain ε_V is zero.

25. Potential Energy in Shear. Relation Between Three Elastic Constants *E*, *G* and μ

Let us calculate the potential energy in shear. For simplicity, face AD of the element is assumed to be fixed (see Fig. 3.5). Then the potential energy will be equal to the work done by force Q acting over face BC:

$$\Pi = \frac{Q\Delta S}{2} = \frac{Q^2 a}{2GF}. \tag{3.14}$$

The unit potential energy will be

$$u = \frac{\Pi}{V} = \frac{Q^2 a}{2GF^2 a} = \frac{\tau^2}{2G} \tag{3.15}$$

$$\left(\text{since } \tau = \frac{Q}{F}, \quad V = Fa\right).$$

On the other hand, the potential energy can be calculated as the work done by the principal normal stresses (Fig. 3.3). From formula (2.53) for a state of plane stress, which is pure shear, we obtain assuming $\sigma_2=0$

$$u=\frac{1}{2E}(\sigma_1^2+\sigma_3^2-2\mu\sigma_1\sigma_3). \tag{3.16}$$

But the principal stresses are $\sigma_1=\tau$, $\sigma_3=-\tau$; therefore

$$u=\frac{\tau^2(1+\mu)}{E}. \tag{3.17}$$

As the magnitude of the energy must not depend on the orientation of the faces of an element, by equating the

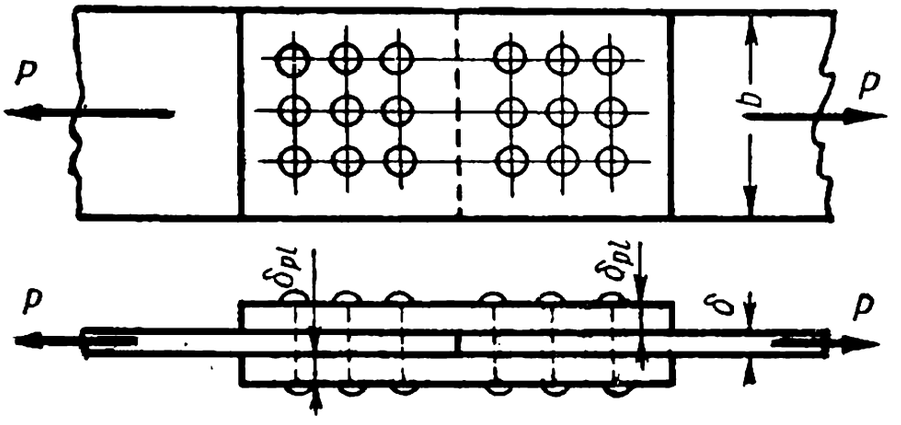


Fig. 3.6

right-hand members of expressions (3.15) and (3.17) we obtain

$$\frac{\tau^2}{2G}=\frac{\tau^2(1+\mu)}{E}.$$

Hence we find the relation between shearing modulus G and the modulus of elasticity of the first kind E:

$$G=\frac{E}{2(1+\mu)}. \tag{3.18}$$

For steel

$$G=\frac{2\times10^6}{2(1+0.3)}\cong8\times10^5\ \text{kg/cm}^2.$$

Example 3.1. Design a rivet joint connecting two sheets of the same cross section $\delta=16$ mm in thickness having two cover plates (Fig. 3.6) if $P=60$ t. Allowable stresses $[\sigma_c]=[\sigma_t]=1{,}600$ kg/cm^2, $[\tau_{sh}]=1{,}000$ kg/cm^2, $[\sigma_{crush}]=3{,}200$ kg/cm^2.

Solution. In this case the rivets are in double shear because in order for a joint to fail there must be a failure in each rivet by shearing along two planes. Take the diameter of a rivet $d=20$ mm. The necessary number of shear planes is found according to formula (3.5):

$$n=\frac{P}{\frac{\pi d^2}{4}[\tau_{sh}]}=\frac{60{,}000\times 4}{3.14\times 2^2\times 1{,}000}=17.4 \text{ shear planes.}$$

Consequently, it is necessary to take 9 rivets. The required number of rivets based on crushing is given by formula (3.8)

$$n'=\frac{P}{\delta d\,[\sigma_{crush}]}=\frac{60{,}000}{1.6\times 2\times 3{,}200}=5.85\approx 6 \text{ rivets.}$$

The design against resistance to shearing has proved to be decisive. We take 9 rivets on each side of the joint in three

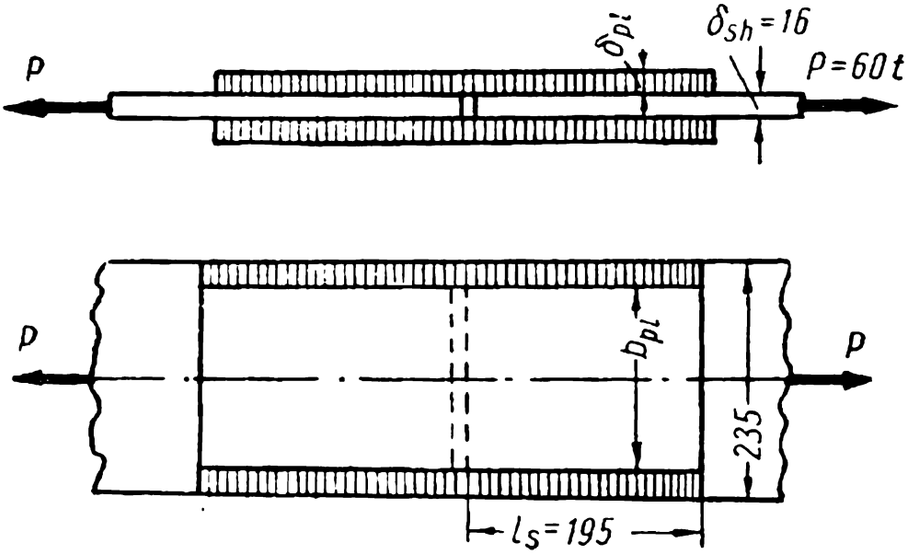


Fig. 3.7

rows with 3 rivets in a row (see Fig. 3.6). Choose the section F of the sheet based on tension:

$$F=\frac{P}{[\sigma_t]}=\frac{60{,}000}{1{,}600}=37.5 \text{ cm}^2.$$

Hence for the thickness $\delta=1.6$ cm we find the width of the sheet

$$b_1=\frac{F}{\delta}=\frac{37.5}{1.6}=23.5 \text{ cm.}$$

The width of the holes $3d=6$ cm must be added to this working width to obtain the total width of the sheet

$b=23.5+6=29.5$ cm. This width is quite enough to take three rivets. (The distance between the centres of rivets is taken to be $3d$.) The thickness δ_{pl} of each cover plate must not be less than half the thickness of the sheet; we take $\delta_{pl}=0.8$ cm.

Example 3.2. Design a welded joint according to the data of the preceding example (Fig. 3.7). The allowable shearing stress for welds is $[\tau_w]=1{,}100$ kg/cm^2.

Solution. In order to leave a place for laying out side fillet welds take the width of the cover plate b_{pl} somewhat narrower than the width of the sheet b_{sh}, i.e., $b_{pl}=$ $=b_{sh}-2\delta=235-32=203$ mm. According to the condition of equal strength the cross-sectional area of the two cover plates must not be less than the cross-sectional area of the sheet, i.e., $2b_{pl}\delta_{pl}\geqslant F_{sh}$. Hence the thickness of the cover plate is

$$\delta_{pl}\geqslant\frac{1.6\times 23.5}{2\times 20.3}=0.92 \text{ cm}.$$

Assume $\delta_{pl}=10$ mm. The necessary working length of the side fillet welds l_s can be determined from the condition [(see formula (3.10)]: $1{,}100\times 4\times 0.7\, l_s\geqslant 60{,}000$. Hence $l_s=$ $=19.5$ cm.

CHAPTER IV

GEOMETRICAL CHARACTERISTICS OF SECTIONS

26. Statical Moment of an Area

In the further study of strength, rigidity and stability we shall have to deal with some geometrical characteristics of a section: statical moments, moments of inertia, section moduli.

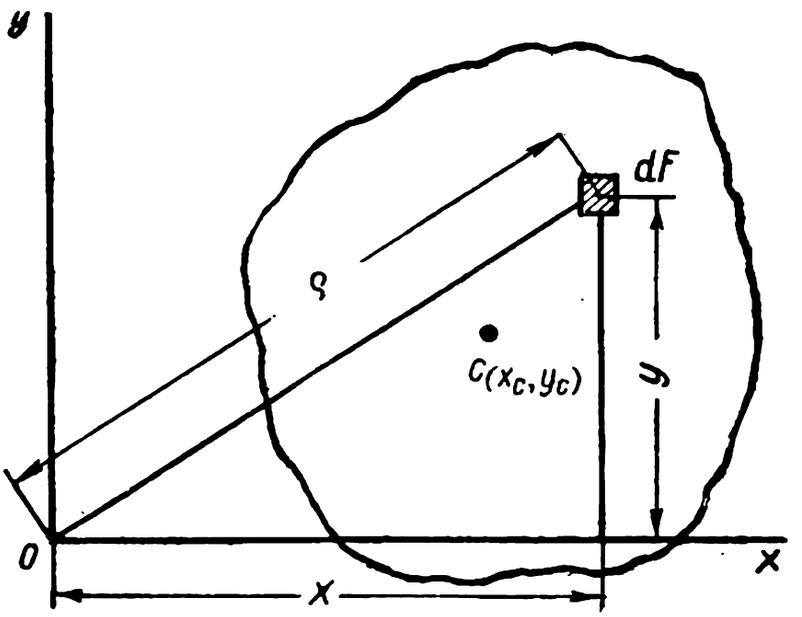


Fig. 4.1

The statical moment S_x of an area with respect to an axis x (Fig. 4.1) is the quantity defined by an integral of the form

$$S_x = \int_F y\,dF, \tag{4.1}$$

where y is the distance of the elementary area dF from the x-axis.

The dimension of the statical moment is the unit of length to the third power, usually cm^3. It may be positive,

negative and, in particular, zero. If an area is identified with a force, integral (4.1) can be regarded as the sum of the moments of the forces dF with respect to the x-axis. According to the theorem of the moment of a resultant known from theoretical mechanics we can write

$$S_x = \int_F y\,dF = Fy_c, \tag{4.2}$$

where F is the area of the whole figure (the resultant) and y_c is the distance of the centroid of the figure from the x-axis. Thus, expression (4.2) yields the formula for determining the ordinate of the centroid

$$y_c = \frac{S_x}{F}. \tag{4.3}$$

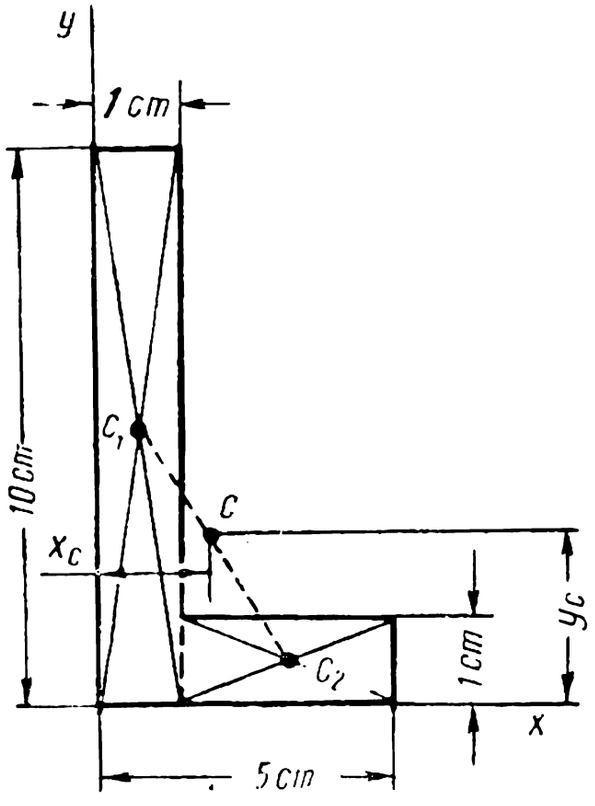


Fig. 4.2

Similarly, the statical moment with respect to the y-axis is

$$S_y = \int_F x\,dF = Fx_c, \tag{4.4}$$

whence

$$x_c = \frac{S_y}{F}. \tag{4.5}$$

It can be seen from these formulas that if the x and y axes pass through the centroid of an area, the statical moment of the area with respect to these axes is equal to zero. We shall refer to such axes in future as *the centroidal axes.*

If a figure can be presented in the form of separate simple figures (squares, triangles, etc.) for which the positions of the centroids are known, the statical moment of the area of the entire figure may be obtained as the sum of the statical moments of the areas of these simple figures. This statement follows directly from the properties of a definite integral.

If a figure has an axis of symmetry, the latter always passes through the centroid of the figure, and therefore

the statical moment of the area of the figure about the axis of symmetry is always zero.

Example 4.1. Determine the position of the centroid of the area (Fig. 4.2).

Solution. Divide the figure into two rectangles. Draw auxiliary x and y axes. According to formulas (4.3) and (4.5) we have

$$x_c = \frac{S_y}{F} = \frac{F_1 x_1 + F_2 x_2}{F_1 + F_2} = \frac{10 \times 1 \times 0.5 + 4 \times 1 \times 3}{10 + 4} = 1.2 \text{ cm},$$

$$y_c = \frac{S_x}{F} = \frac{F_1 y_1 + F_2 y_2}{F_1 + F_2} = \frac{10 \times 1 \times 5 + 4 \times 1 \times 0.5}{14} = 3.7 \text{ cm}.$$

These co-ordinates determine the location of the centroid C of the area.

27. Moment of Inertia of an Area

The axial (or equatorial) moment of inertia of the cross-sectional area is a geometrical characteristic of the section numerically equal to the integral:

$$J_x = \int_F y^2 \, dF. \tag{4.6}$$

Similarly, with respect to the y-axis:

$$J_y = \int_F x^2 \, dF,$$

where y is the distance of the elementary area dF from the x-axis (see Fig. 4.1), and x is the distance of the elementary area dF from the y-axis.

The polar moment of inertia of the area of a figure is a geometrical characteristic of the section defined by an integral of the form

$$J_p = \int_F \rho^2 dF, \tag{4.7}$$

where ρ is the distance of the elementary area dF from the point (pole) (see Fig. 4.1) about which the polar moment of inertia is taken.

The axial and polar moments of inertia are essentially positive quantities,

In fact, whatever be the sign of the co-ordinate of an arbitrary area, the corresponding term is positive as it includes the square of that co-ordinate.

The product of inertia of the area of a figure is a geometrical characteristic of the section defined by an integral of the form

$$J_{xy} = \int_F xy\, dF, \tag{4.8}$$

where x, y are the distances of the elementary area dF from the x and y axes.

The dimension of all the moments of inertia is the unit of length to the fourth power (usually cm^4).

The product of inertia may be positive, negative and, in a particular case, zero.

If orthogonal axes x and y, or one of them, are axes of symmetry of a figure, the product of inertia with respect to such axes is equal to zero. Indeed, where a symmetrical figure is concerned, it is always possible to find two elements of its area (Fig. 4.3) which have the same ordinates y and abscissas that are equal but opposite in sign. Summing the products $xydF$ for such elements, i. e., computing integral (4.8), we get zero.

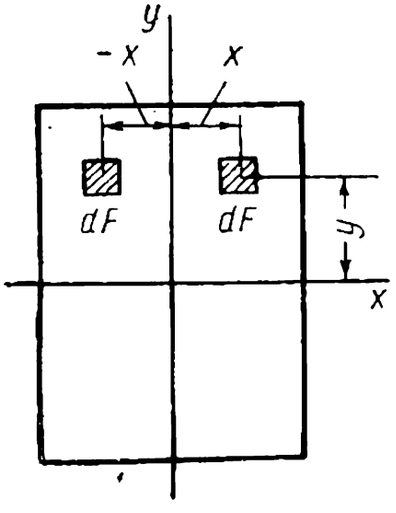


Fig. 4.3

It is easy to prove that the polar moment of inertia with respect to any point is equal to the sum of the axial moments of inertia with respect to two perpendicular axes passing through that point.

In fact, it can be seen from Fig. 4.1 that $\rho^2 = x^2 + y^2$. Substituting this value of ρ^2 in expression (4.7) we have

$$J_p = \int_F \rho^2\, dF = \int_F (x^2 + y^2)\, dF = \int_F x^2\, dF + \int_F y^2\, dF.$$

Therefore

$$J_p = J_x + J_y. \tag{4.9}$$

28. Relation Between Moments of Inertia with Respect to Parallel Axes

Let us determine the moment of inertia of a figure with respect to any axis x_1 (Fig. 4.4).

Let x_0 be a centroidal axis and let the moment of inertia J_{x_0} be known. It is seen from the drawing that $y_1 = a + y$. Therefore

$$J_{x_1} = \int_F (a+y)^2 dF = a^2 \int_F dF + 2a \int_F y\, dF + \int_F y^2 dF.$$

The first integral gives the area of the section.

The second integral which represents the statical moment of the area about the centroidal axis x_0 is equal to zero.

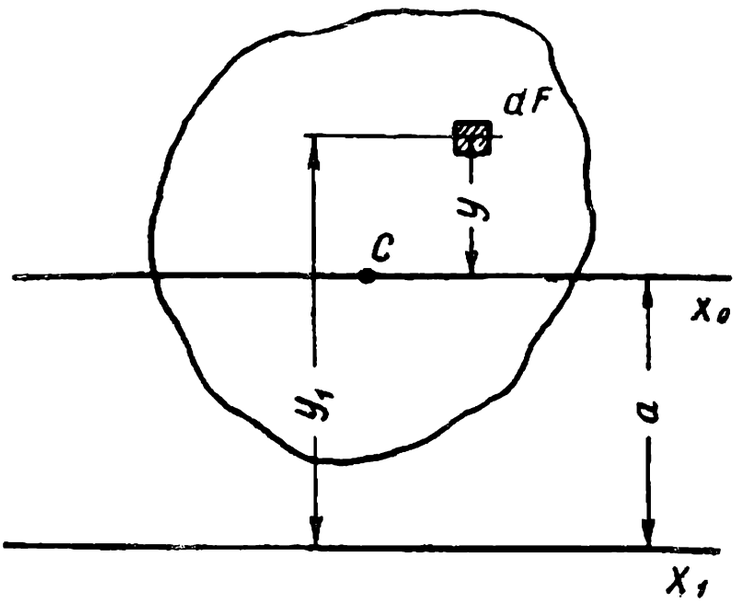


Fig. 4.4

The third integral represents the moment of inertia J_{x_1} of the area of the figure with respect to the x_0-axis.

Thus

$$J_{x_1} = J_{x_0} + Fa^2. \qquad (4.10)$$

The moment of inertia with respect to any axis is equal to the moment of inertia with respect to a centroidal axis parallel to the given axis plus the product of the area of the figure and the square of the distance between the two axes.

It can be seen from formula (4.10) that the moment of inertia with respect to a centroidal axis is less than the

moment of inertia with respect to any non-centroidal axis parallel to the former.

The moment of inertia with respect to a centroidal axis is called *the centroidal moment of inertia.*

29. Moments of Inertia of Simple Sections

1. **Rectangle** (Fig. 4.5*a*). Let us calculate the moment of inertia of a section with respect to an axis x_0 passing through the centroid.

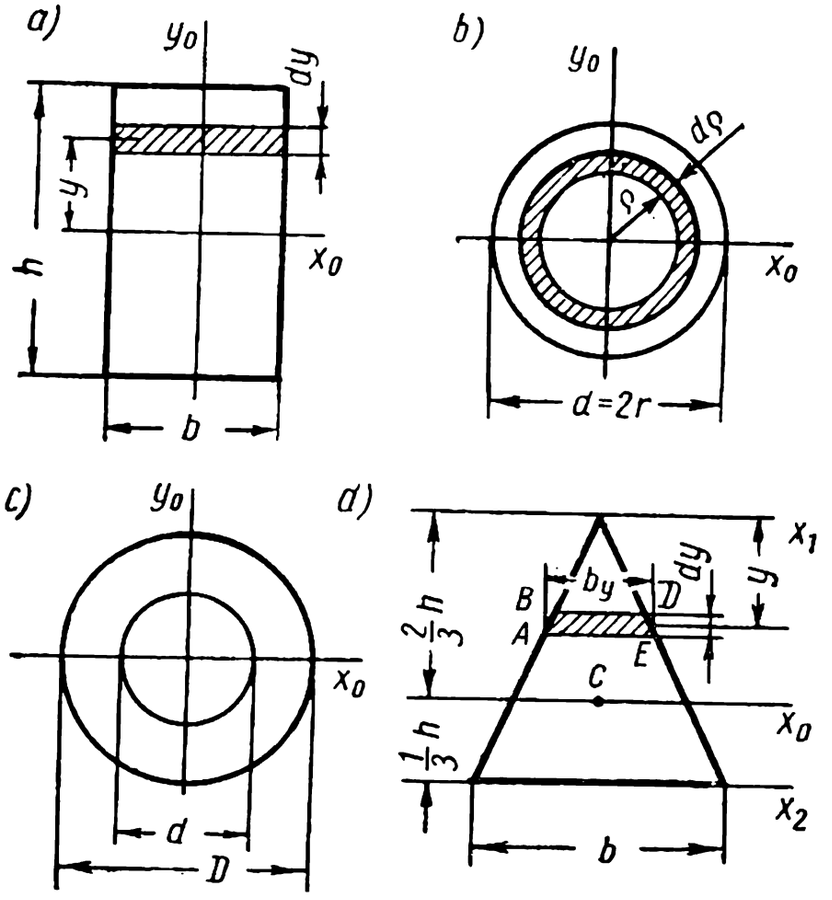


Fig. 4.5

The element of area dF will be taken as an infinitely thin layer $dF = b\,dy$.

Then

$$J_{x_1} = b \int_{-\frac{h}{2}}^{+\frac{h}{2}} y^2\,dy = \frac{bh^3}{12}.$$

Thus

$$J_{x_0} = \frac{bh^3}{12}. \qquad (4.11)$$

Likewise

$$J_{y_0} = \frac{hb^3}{12}. \tag{4.12}$$

2. **Circle** (Fig. 4.5*b*). Let us first determine the polar moment of inertia with respect to the centre of the circle:

$$J_p = \int_F \rho^2 dF.$$

The elementary area dF will be taken as an infinitely thin ring of width $d\rho$:

$$dF = 2\pi\rho\, d\rho,$$

then

$$J_p = 2\pi \int_0^r \rho^3 d\rho = \frac{\pi r^4}{2}.$$

Therefore

$$J_p = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \approx 0.1\, d^4. \tag{4.13}$$

Now we can easily find J_{x_0}. For a circle, according to formula (4.9), we have $J_p = 2J_{x_0} = 2J_{y_0}$. Hence

$$J_{x_0} = J_{y_0} = \frac{J_p}{2} = \frac{\pi r^4}{4} = \frac{\pi d^4}{64} \approx 0.05\, d^4. \tag{4.14}$$

3. **Ring** (Fig. 4.5*c*). The axial moment of inertia in this case is equal to the difference between the moments of inertia of the outside and inside circles:

$$J_{x_0} = J_{y_0} = \frac{\pi D^4}{64} - \frac{\pi d^4}{64} = 0.05\, D^4 (1 - c^4), \tag{4.15}$$

where

$$c = \frac{d}{D}.$$

Similarly, the polar moment of inertia

$$J_p = \frac{\pi D^4}{32} - \frac{\pi d^4}{32} = 0.1\, D^4 (1 - c^4). \tag{4.16}$$

4. **Triangle** (Fig. 4.5*d*). Determine the moment of inertia with respect to an axis x_1 parallel to the base and passing through the vertex of the triangle:

$$J_{x_1} = \int_F y^2\, dF.$$

The element of area dF will be selected as an infinitely thin trapezoid $ABDE$ whose area can be considered equal to the area of the rectangle:

$$dF = b_y\, dy,$$

where b_y is the length of the rectangle. From similar triangles we can easily obtain:

$$b_y = \frac{y}{h}\, b,$$

then

$$J_{x_1} = \frac{b}{h} \int_0^h y^3 dy = \frac{bh^3}{4}. \tag{4.17}$$

Determine the moment of inertia with respect to a centroidal axis by using formula (4.10):

$$J_{x_0} = J_{x_1} - Fa^2 = \frac{bh^3}{4} - \frac{bh}{2}\left(\frac{2}{3}\,h\right)^2 = \frac{bh^3}{36}. \tag{4.18}$$

Determine the moment of inertia with respect to an axis coinciding with the base:

$$J_{x_2} = J_{x_0} + Fa^2 = \frac{bh^3}{36} + \frac{bh}{2}\left(\frac{1}{3}\,h\right)^2 = \frac{bh^3}{12}. \tag{4.19}$$

30. Moments of Inertia of Composite Figures

The moment of inertia of a composite figure is equal to the sum of the moments of inertia of its separate parts:

$$J_x = J_x^{\mathrm{I}} + J_x^{\mathrm{II}} + J_x^{\mathrm{III}} + \ldots \tag{4.20}$$

This follows directly from the properties of a definite integral:

$$\int_F y^2\, dF = \int_{F_1} y^2\, dF + \int_{F_2} y^2\, dF + \ldots,$$

where

$$F = F_1 + F_2 + \ldots$$

Thus, to determine the moment of inertia of a composite figure it is necessary to divide this figure into a number of simple figures, calculate the moments of inertia of these figures and then sum up these moments of inertia.

The foregoing theorem is valid for the product of inertia as well.

The moments of inertia of rolled sections (I-sections, channels, angles, etc.) are given in tables on profile sections.

31. Change of Moments of Inertia with Rotation of Axes

Let us find the relation between the moments of inertia with respect to the axes x, y and the moments of inertia with respect to the axes x_1, y_1 rotated through an angle α

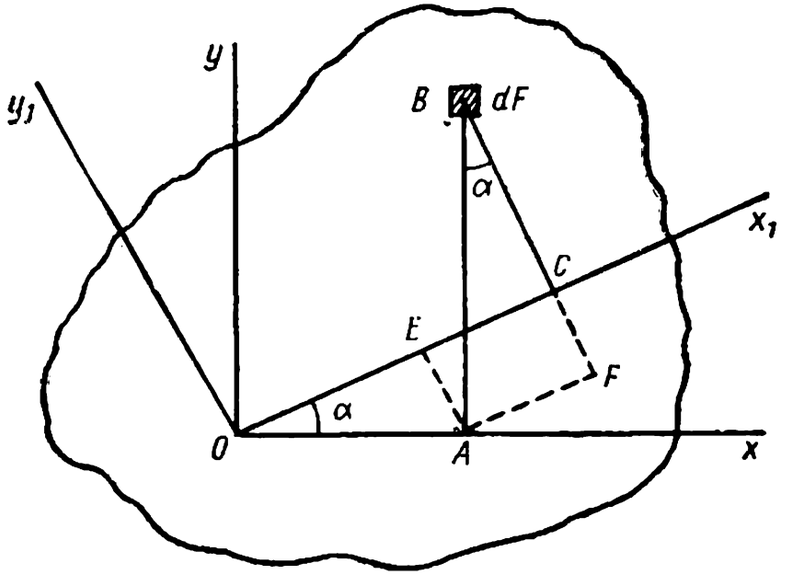


Fig. 4.6

(Fig. 4.6). Let $J_x > J_y$ and let the angle α be taken as positive in a counter-clockwise direction from the x-axis.

To solve the above problem let us find the relation between the co-ordinates of the elementary area dF with respect to the given and rotated axes. It follows from the drawing that

$$x_1 = \overline{OC} = \overline{OE} + \overline{EC} = \overline{OE} + \overline{AF} = \overline{OA}\cos\alpha + AB\sin\alpha = \\ = x\cos\alpha + y\sin\alpha, \qquad (4.21)$$

$$y_1 = \overline{BC} = \overline{BF} - \overline{AE} = y\cos\alpha - x\sin\alpha. \qquad (4.22)$$

Now we shall determine the moments of inertia with respect to the x_1 and y_1 axes:

$$J_{x_1} = \int_F y_1^2 dF = \int_F (y \cos \alpha - x \sin \alpha)^2 dF =$$

$$= \int_F y^2 \cos^2 \alpha \, dF - 2 \int_F xy \sin \alpha \cos \alpha \, dF + \int_F x^2 \sin^2 \alpha \, dF$$

or

$$J_{x_1} = J_x \cos^2 \alpha + J_y \sin^2 \alpha - 2J_{xy} \sin \alpha \cos \alpha. \quad (4.23)$$

Likewise

$$J_{y_1} = \int_F (x \cos \alpha + y \sin \alpha)^2 dF =$$

$$= J_x \sin^2 \alpha + J_y \cos^2 \alpha + 2J_{xy} \sin \alpha \cos \alpha, \quad (4.24)$$

$$J_{x_1 y_1} = \int_F (x \cos \alpha + y \sin \alpha)(y \cos \alpha - x \sin \alpha) dF =$$

$$= \frac{J_x \sin 2\alpha}{2} - \frac{J_y \sin 2\alpha}{2} + J_{xy} \cos 2\alpha. \quad (4.25)$$

By adding (4.23) and (4.24) we obtain

$$J_{x_1} + J_{y_1} = J_x + J_y = J_p. \quad (4.26)$$

By subtracting (4.24) from (4.23) we have

$$J_{x_1} - J_{y_1} = (J_x - J_y) \cos 2\alpha - 2J_{xy} \sin 2\alpha. \quad (4.27)$$

Formula (4.26) shows that *the sum of the moments of inertia with respect to any perpendicular axes does not change when the latter rotate.*

Formula (4.27) may be useful for calculating the product of inertia with respect to the axes x, y if the axial moments of inertia with respect to the axes x, y and x_1, y_1 are known.

32. Principal Axes of Inertia and Principal Moments of Inertia

With a change in the angle α the quantities J_{x_1}, J_{y_1} and $J_{x_1 y_1}$ vary. Let us find the value of the angle which makes J_{x_1} and J_{y_1} have extremal values by taking the first deriv-

ative of J_{x_1} or J_{y_1} with respect to α and setting it equal to zero:

$$\frac{dJ_{x_1}}{d\alpha} = -2J_x \cos\alpha_0 \sin\alpha_0 + 2J_y \sin\alpha_0 \cos\alpha_0 - 2J_{xy} \cos 2\alpha_0 = 0$$

or

$$(J_y - J_x) \sin 2\alpha_0 - 2J_{xy} \cos 2\alpha_0 = 0,$$

whence

$$\tan 2\alpha_0 = \frac{2J_{xy}}{J_y - J_x}. \tag{4.28}$$

This formula determines the position of two axes about one of which the axial moment of inertia is a maximum and about the other a minimum.

Such axes are called the *principal* axes. Moments of inertia with respect to the principal axes are called *the principal moments of inertia.*

The magnitude of the principal moments of inertia may be found from formula (4.23) or (4.24) by substituting the value of α_0 from formula (4.28) and by using the known trigonometric identities for functions of double angles (see Section 19).

After some transformations we obtain the following formula for determining *the principal moments of inertia*:

$$J_{\substack{\max \\ \min}} = \frac{J_x + J_y}{2} \pm \frac{1}{2} \sqrt{(J_x - J_y)^2 + 4J_{xy}^2}. \tag{4.29}$$

This formula is identical in its structure with (2.36), for principal stresses.

Investigating the second derivative $\frac{d^2J_{x_1}}{d\alpha^2}$ it may be established that in the present case the maximum moment of inertia $J_{\max}$ occurs with respect to the principal axis making an angle α_0 with the x-axis, and the minimum moment of inertia, with respect to the other perpendicular axis. In most cases there is no need for such an investigation since it is evident from the configuration of the section which of the principal axes corresponds to the maximum of the moment of inertia. In addition to formula (4.29) use can also be made of formulas (4.23) and (4.24) for the determination of the principal moments of inertia. The question of the principal axes with respect to which the maximum and minimum moments of inertia occur is then decided by itself.

Let us now show that the product of inertia with respect to the principal axes is zero.

In reality, setting $J_{x_1y_1}$ (formula 4.25) equal to zero we have

$$\frac{J_x - J_y}{2} \sin 2\alpha_0 + J_{xy} \cos 2\alpha_0 = 0,$$

from which for tan $2\alpha_0$ we again obtain formula (4.28).

Thus, the principal axes of inertia are defined as axes possessing the following properties:

1. *The product of inertia is zero with respect to these axes.*

2. *The moments of inertia with respect to the principal axes have extremal values (a maximum with respect to one axis, a minimum with respect to the other).*

The principal axes passing through the centroid of the section are called the principal centroidal axes of inertia.

In many cases the position of the principal centroidal axes can readily be determined. If a figure has an axis of symmetry, it is one of the principal centroidal axes, the other passing through the centroid of a section perpendicular to the first one. This statement follows from the fact that the product of inertia is equal to zero with respect to an axis of symmetry and any axis perpendicular to it.

By using formulas (4.23) through (4.25) it can be shown that if two principal centroidal moments of inertia of a section are equal, then any centroidal axis of this section is principal and all the principal centroidal moments of inertia are the same (circle, square, hexagon, equilateral triangle).

In fact, suppose that for a certain section the x and y axes are principal centroidal axes and, in addition, $J_x = J_y$. Then from formulas (4.23) and (4.24) we ascertain that $J_x = J_y = J_{x_1} = J_{y_1}$, and from formula (4.25) we see that $J_{x_1y_1} = 0$, i.e., any axes x_1 and y_1 are principal centroidal axes of inertia of such a figure.

33. Relation Between Products of Inertia with Respect to Two Sets of Parallel Axes

Let the axes x_0, y_0 be centroidal axes (Fig. 4.7) and let the moment of inertia $J_{x_0y_0}$ be given. Let us find the product of inertia with respect to axes x_1, y_1. It is seen from

the drawing that

$$x_1 = x_0 + b,$$

$$y_1 = y_0 + a.$$

Therefore

$$J_{x_1y_1} = \int_F x_1 y_1 \, dF = \int_F (x_0 + b)(y_0 + a) \, dF =$$

$$= \int_F x_0 y_0 \, dF + b \int_F y_0 \, dF + a \int_F x_0 \, dF + \int_F ab \, dF$$

or

$$J_{x_1y_1} = J_{x_0y_0} + Fab. \tag{4.30}$$

The second and third integrals in the right-hand member of equality (4.30) which represent the statical moments with respect to the centroidal axes are equal to zero.

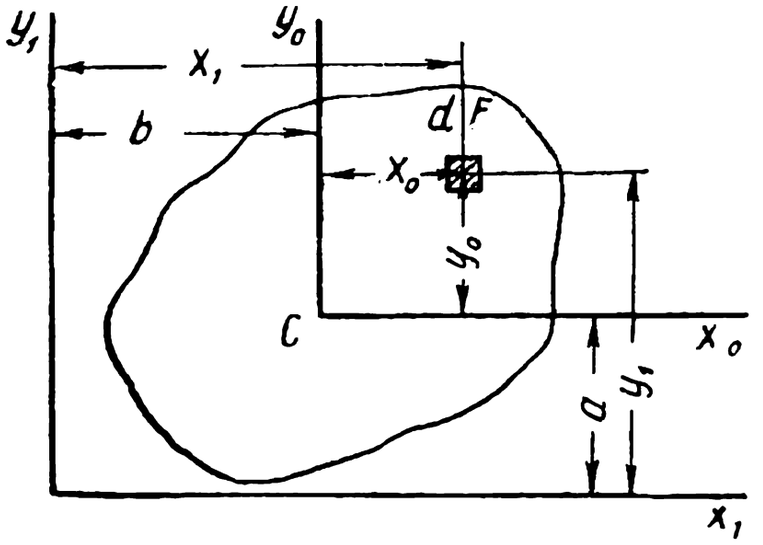


Fig. 4.7

Thus, *the product of inertia with respect to any set of perpendicular axes parallel to centroidal axes is equal to the product of inertia with respect to these centroidal axes plus the product of the area of the figure and the co-ordinates of its centroid with respect to the new axes.*

If the axes x_0, y_0 are the centroidal principal axes, then $J_{x_0y_0} = 0$ with respect to these axes and formula (4.30) is simplified:

$$J_{x_1y_1} = Fab. \tag{4.31}$$

For a composite figure made of n simple figures

$$J_{x_1y_1} = \sum_1^n F_i a_i b_i \qquad (4.32)$$

(provided that the centroidal axes of each figure are the principal axes).

Example 4.2. Calculate the product of inertia of an angle section 100×100×10 mm with respect to the centroidal x_1 and y_1 axes (Fig. 4.8).

Solution. We shall make use of formula (4.25) and determine the product of inertia by means of the moments of inertia with respect to the principal centroidal axes x_0 and y_0, ascertained from tables on profile sections:

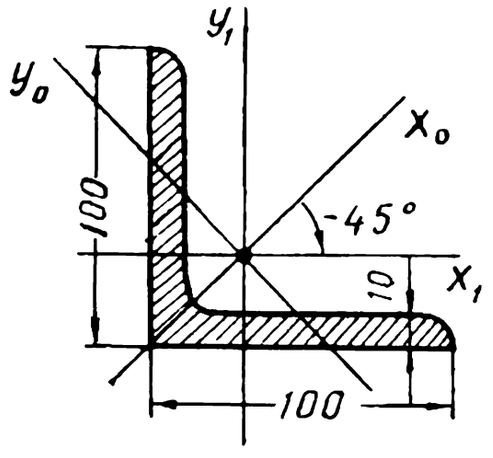


Fig. 4.8

$$J_{x_0} = 284 \text{ cm}^4,$$

$$J_{y_0} = 74.1 \text{ cm}^4,$$

$$J_{x_1y_1} = \frac{(J_{x_0} - J_{y_0}) \sin 2\alpha}{2} + J_{x_0y_0} \cos 2\alpha.$$

Since the x_0 and y_0 axes are principal centroidal axes (the x_0-axis is an axis of symmetry of the figure), the moment of inertia $J_{x_0y_0}$ is equal to zero. The angle $\alpha = -45°$ since the x_1 and y_1 axes with respect to which the product of inertia is being calculated are rotated clockwise about the x_0 and y_0 axes.

Therefore,

$$J_{x_1y_1} = \frac{284 - 74.1}{2}(-1) = -104.95 \text{ cm}^4.$$

Example 4.3. Calculate the principal centroidal moments of inertia of the area of a section (Fig. 4.9).

Table 4.1

Figure	Area of section, F, cm²	Position of centroid, z_0, cm	Moments of inertia with respect to own centroidal axes, cm⁴	
			horizontal	vertical
Channel No. 20ª	28.83	2.01	1780.4	128
Angle 100×100×10	19.2	2.83	179	179

Solution. 1. Determine the co-ordinates of the centroid by drawing auxiliary axes x_1 and y_1 and dividing the section into two figures, a channel (I) and an angle (II), for which all the necessary data are available in tables on profile sections (Table 4.1).

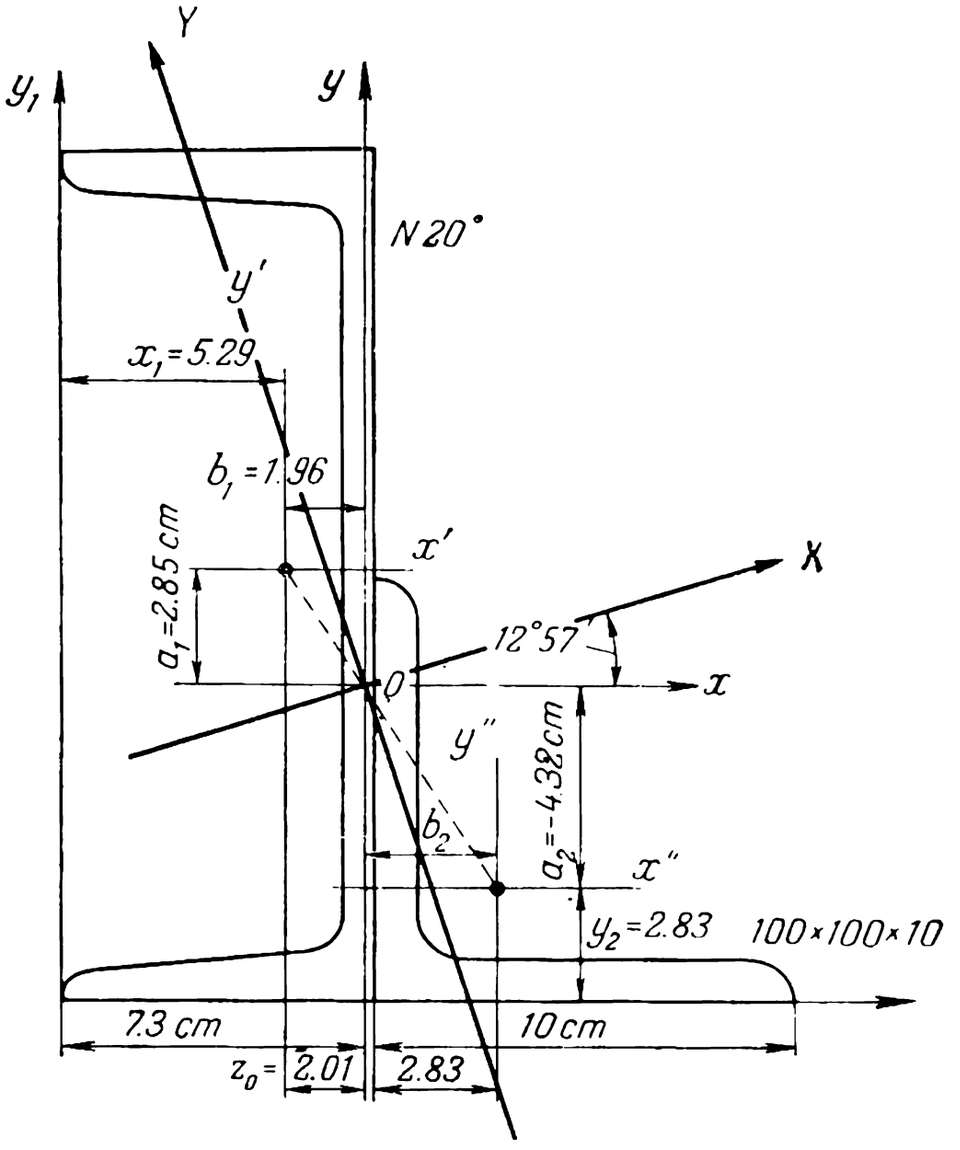


Fig. 4.9

The co-ordinates of the centroid of the section are determined by the formulas

$$x_0 = \frac{F_1 x_1 + F_2 x_2}{F_1 + F_2},$$

$$y_0 = \frac{F_1 y_1 + F_2 y_2}{F_1 + F_2},$$

where

F_1 = area of first figure (channel);

x_1 = distance from y_1-axis to centroid of channel.

Here $x_1 = 7.3 - 2.01 = 5.29$ cm, where 7.3 cm is the width of the flange of the channel, and $z_0 = 2.01$ cm is the distance from the centroid of the channel to the web;

F_2 = cross-sectional area of angle;

x_2 = distance from centroid of angle to y_1-axis.

Here $x_2 = 7.3 + 2.83 = 10.13$ cm. In the present example $y_1 = 10$ cm, $y_2 = 2.83$ cm (see Fig. 4.9).

Substituting the numerical values, we obtain

$$x_0 = \frac{28.83 \cdot 5.29 + 19.2 \cdot 10.13}{28.83 + 19.2} = 7.25 \text{ cm},$$

$$y_0 = \frac{28.83 \cdot 10 + 19.2 \cdot 2.83}{28.83 + 19.2} = 7.15 \text{ cm}.$$

By the use of these data we plot a point O, the centroid of the section, and draw auxiliary centroidal axes x and y. Then O must lie on the line joining the centroids of the channel and the angle.

2. Calculate the moments of inertia with respect to the x and y axes

$$J_x = J_x^{\mathrm{I}} + J_x^{\mathrm{II}},$$

$$J_y = J_y^{\mathrm{I}} + J_y^{\mathrm{II}}.$$

For the calculation of the moment of inertia of the channel J_x^{I} with respect to the x-axis we use formula (4.10)

$$J_x^{\mathrm{I}} = 1780.4 + 28.83\,(+2.85)^2 = 2015.4 \text{ cm}^4,$$

where

$J_{x'} = 1780.4$ cm^4 = moment of inertia of channel with respect to own centroidal x'-axis;

$F = 28.83$ cm^2 = cross-sectional area of channel;

$a_1 = 10 - 7.15 = 2.85$ cm = distance from x-axis to centroid of channel, taken with plus sign because centroid of channel is above x-axis.

In a similar way the moment of inertia of the angle with respect to the x-axis is

$$J_x^{\mathrm{II}} = 179 + 19.2\,(-4.32)^2 = 539 \text{ cm}^4,$$

where

$J_{x''} = 179$ cm^4 = moment of inertia of angle with respect to own centroidal x''-axis;

$a_2 = 2.83 - 7.15 = -4.32$ cm = distance from x-axis to centroid of angle. The minus sign is taken because the centroid of the angle is below the x-axis.

The total moment of inertia of the section with respect to the x-axis is

$$J_x = 2015.4 + 539 = 2554.4 \text{ cm}^4.$$

The moment of inertia of the section with respect to the y-axis is calculated in exactly the same way.

For the channel

$$J_y^{\text{I}} = 128 + 28.3(-1.96)^2 = 239 \text{ cm}^4,$$

where

$b_1 = 5.29 - 7.25 = -1.96$ cm—distance from y-axis to centroid of channel. The minus sign is taken because the centroid of the channel is located to the left of the y-axis.

For the angle

$$J_y^{\text{II}} = 179 + 19.2(2.88)^2 = 238.5 \text{ cm}^4,$$

where

$b_2 = 7.3 + 2.83 - 7.25 = 2.88$ cm—distance from y-axis to centroid of angle.

The total moment of inertia of the section with respect to the y-axis is

$$J_y = 239 + 238.5 = 477.5 \text{ cm}^4.$$

Calculate the product of inertia of the section with respect to the x and y axes.

To do this we make use of formula (4.30). Since the channel has the horizontal axis of symmetry x', the centroidal x' and y' axes of the channel are principal axes, and therefore the first term in formula (4.30) is zero for the channel.

The centroidal axes of the angle parallel to the x and y axes, i.e., the x'' and y'' axes are not principal axes, therefore the first term in formula (4.30) is not zero for the angle. It should be calculated in the same way as in Example 4.2. The value of the product of inertia of a similar angle with respect to the x'' and y'' axes was found to be given by $J_{x''y''} = -104.95 \text{ cm}^4$.

Consequently the product of inertia of the whole section is

$$J_{xy} = 28.83\,(2.85)\,(-1.96) - 104.95 + 19.2\,(-4.32)\,(2.88) =$$
$$= -160.5 - 104.95 - 239 = -504.45 \text{ cm}^4.$$

3. Determine the position of the principal centroidal axes by means of formula (4.28)

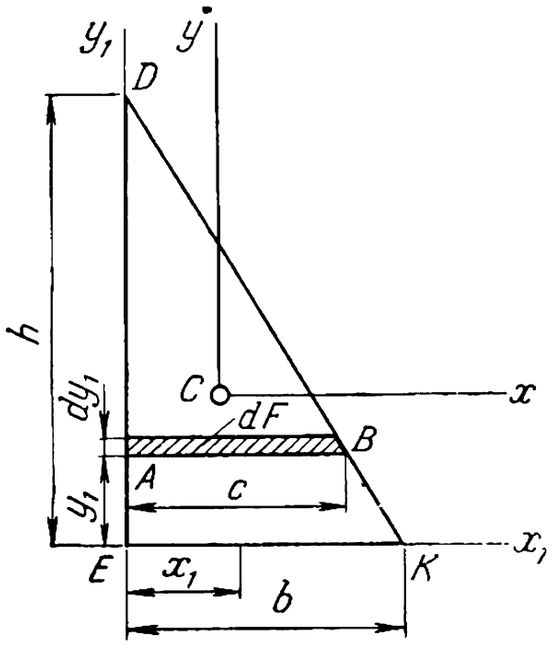


Fig. 4.10

$$\tan 2\alpha_0 = \frac{2J_{xy}}{J_y - J_x} =$$
$$= \frac{2\,(-504.45)}{477.5 - 2554.4} = 0.485,$$
$$2\alpha_0 = 25^\circ 54', \quad \alpha_0 = 12^\circ 57'.$$

Lay off this angle (in a counter-clockwise direction) and draw the principal centroidal axes X and Y. If $\tan 2\alpha_0$ and α_0 came out negative, the principal centroidal axes would be rotated about the x and y axes in a clockwise direction.

4. Calculate the principal centroidal moments of inertia by using formula (4.29)

$$J_{\substack{\max\\\min}} = \frac{2554.4 + 477.5}{2} \pm \frac{1}{2}\sqrt{(2554.4 - 477.5)^2 + 4 \cdot 504.45^2} =$$
$$= 1515.95 \pm 1155.$$

Hence

$$J_{\max} = 2670.95 \text{ cm}^4; \qquad J_{\min} = 360.95 \text{ cm}^4.$$

Since $J_x > J_y$, $J_{\max}$ is about the principal X-axis, and $J_{\min}$ is about the Y-axis.

For verification we use relation (4.26)

$$J_x + J_y = J_{\max} + J_{\min}.$$

In the present case this equality holds

$$2554.4 + 477.5 = 3031.9 \text{ cm}^4 = 2670.95 + 360.95 = 3031.9 \text{ cm}^4.$$

Example 4.4. Calculate the product of inertia of a right triangle with respect to axes coinciding with its legs (Fig. 4.10).

Solution.

The product of inertia is determined by the formula

$$J_{x_1y_1} = \int_F x_1 y_1 \, dF.$$

An element of area is taken to be the area of a strip AB

$$dF = c \, dy_1.$$

The abscissa x_1 of the centroid of the strip AB is

$$x_1 = \frac{c}{2}.$$

From the similar triangles DEK and DAB it follows that

$$c = \frac{b}{h}(h - y_1).$$

Thus

$$J_{x_1y_1} = \frac{b^2}{2h^2} \int_0^h (h - y_1)^2 y_1 \, dy.$$

or

$$J_{x_1y_1} = \frac{b^2h^2}{24}.$$

To determine the product of inertia with respect to the centroidal axes we use formula (4.30)

$$J_{xy} = J_{x_1y_1} - F \frac{h}{3} \frac{b}{3},$$

or

$$J_{xy} = -\frac{b^2h^2}{72}.$$

As can be seen, the product of inertia with respect to the centroidal axes is negative.

CHAPTER V

TORSION

34. Construction of Twisting Moment Diagrams

A bar (shaft) undergoes torsion if at its cross sections twisting moments occur, i.e., moments lying in the plane of the section. These internal twisting moments M_t are generally produced by external moments (Fig. 5.1). External moments are transmitted to a shaft, as a rule, at the points where pulleys, gears, etc., are keyed to it.

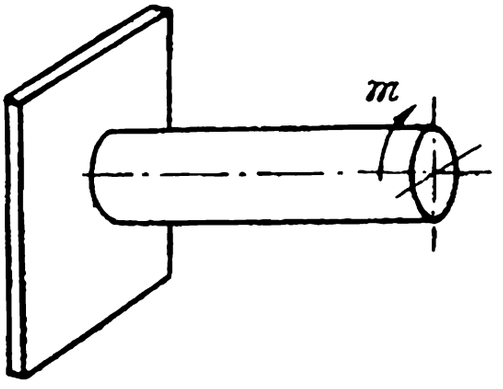


Fig. 5.1

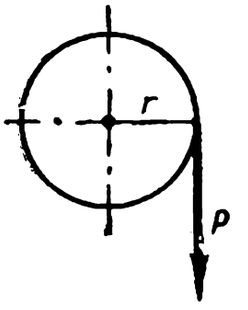


Fig. 5.2

However, a transverse load displaced with respect to the axis of the bar also causes internal twisting moments (Fig. 5.2) but in this case at the cross sections of the bar in addition to twisting moments other internal forces arise as well—shearing forces and bending moments.

The present chapter is concerned with torsion analysis under static loads. The design of shafts under the action of twisting moments variable in time (fatigue analysis) is discussed in Chapter XII.

Instead of an axonometric representation we shall employ mostly a plane representation, as being simpler. External torsional moments and internal twisting moments will be represented as a line with a small circle at either

end of it. In one of them we shall put a point denoting the beginning of an arrow (towards us), in the other a cross denoting the end of an arrow directed away from us (Fig. 5.3).

For determining twisting moments M_t produced at the cross sections of a bar by external torsional moments or a transverse load we shall employ the method of sections. Imagine the bar cut (Fig. 5.3), for instance, across *a-a*, remove one portion of the bar, in this case the left one, and consider the equilibrium of the remaining right portion.

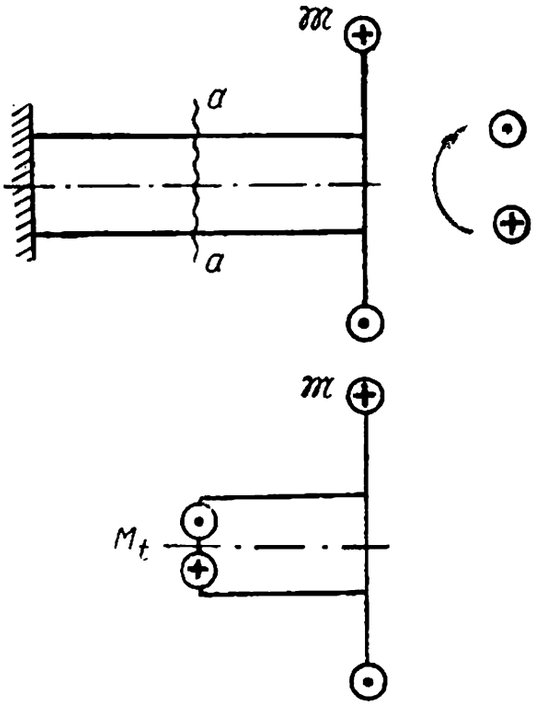


Fig. 5.3

Replace the interaction of the portions of the bar by a twisting moment M_t balancing the external moment $\mathfrak{M}$. In order for the cut portion to be in equilibrium it is necessary that the algebraic sum of all the moments acting on it should be equal to zero.* Hence in the case being considered we obtain $M_t = \mathfrak{M}$. If several external moments act on the cut portion, it can be proved in the same way that *the twisting moment at the section is numerically equal to the algebraic sum of the external torsional moments acting to one side of the section.*

Twisting moment diagrams are constructed to give a visual representation of the nature of distribution and the magnitude of twisting moments along the length of the bar. The construction of these diagrams is similar to the construction of longitudinal force diagrams in tension or compression. In constructing these diagrams it is necessary to decide upon the sign rule. There is no conventional sign rule for twisting moments. Any may be used. The important thing is to keep to the rule decided upon throughout the

* The equilibrium condition for a system of couples acting in parallel planes.

diagram. Let us adopt the following rule for the sign (Fig. 5.4): the twisting moment at the section a-a is considered to be positive if the external moment tends to rotate

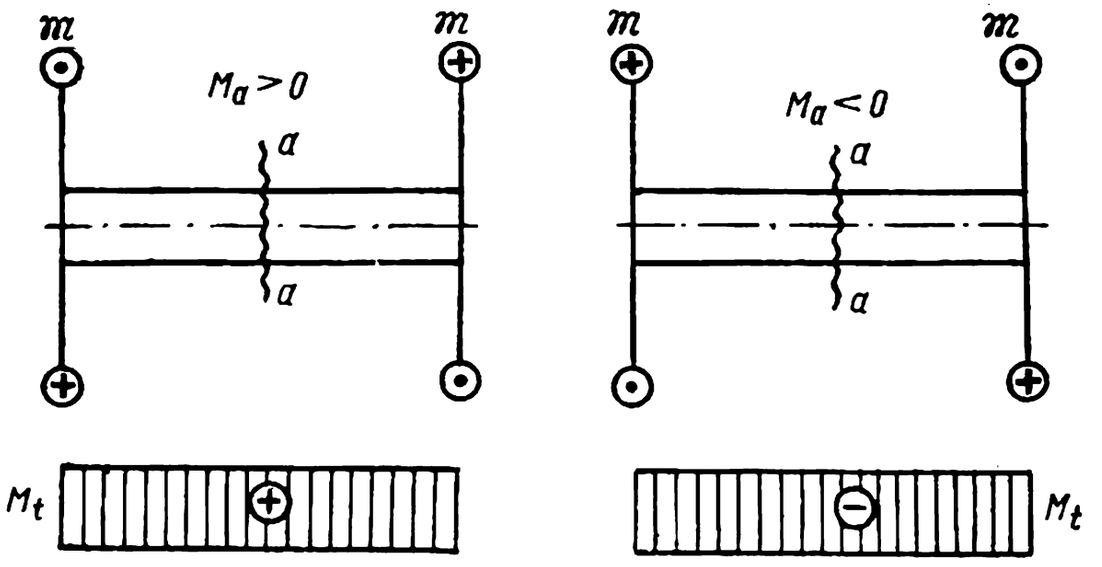


Fig. 5.4

the cut portion counter-clockwise when viewed from the side of the section. If, however, the external moment tends to rotate the cut portion clockwise (when viewed from the

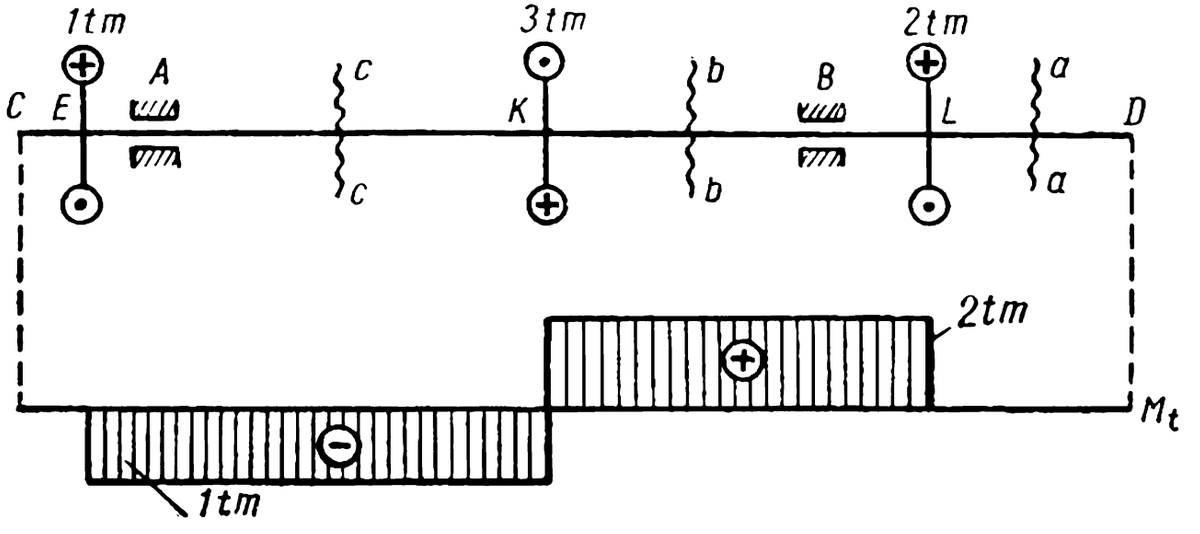


Fig. 5.5

side of the section) the twisting moment at the section is considered to be negative.

Let us illustrate the construction of a twisting moment diagram by the following example (Fig. 5.5): consider a shaft CD being supported by bearings A and B and held in equilibrium under the action of moments applied at sec-

tions E, K and L. Making a section a-a somewhere in the portion DL and considering the equilibrium of the right-hand cut part, we find that $M_t' = 0$. If we make now a section b-b at any place of the portion LK, from the condition of equilibrium of the part to the right of the section we obtain

$$M_t'' = 2 \text{ tm}.$$

The moment is considered positive in accordance with the rule adopted for the sign. Making a section c-c in the portion KE, from the condition of equilibrium of the right-hand part we have

$$2 - 3 - M_t''' = 0,$$

whence

$$M_t''' = -1 \text{ tm}.$$

The diagram thus obtained has the shape of two rectangles. It is important to note that at the places of application of external moments the ordinates of the diagram change stepwise by the amount of the external moment applied here.

If transverse loads are given which cause the bar to twist (see Fig. 5.2), external torsional moments produced by these forces are calculated first. In the case represented in Fig. 5.2 the external torsional moment of the force P is $\mathfrak{M} = Pr$. After determining external moments, internal twisting moments are determined and diagrams are constructed as described above.

35. Determination of Stresses in Bars of Circular Cross Section

The twisting moments which have been under consideration represent merely the resultant internal forces. Actually at a cross section of a twisted bar there act continuously distributed internal tangential forces to the determination of which we now proceed.

Let us first of all examine some experimental results. If a rectangular network of lines is marked out on the surface of a bar of circular cross section the following will be found after deformation (Fig. 5.6):

(1) the rectangular network will have become one composed of parallelograms, an indication of the presence of shearing stresses on the transverse sections of the bar, and by virtue of the law of conjugate shearing stresses on its longitudinal sections as well;

(2) the distances between circles, for instance between *I* and *II*, will not change. Neither will the length of the bar or its diameter change. This proves the absence of normal stresses on transverse and longitudinal sections of the bar.

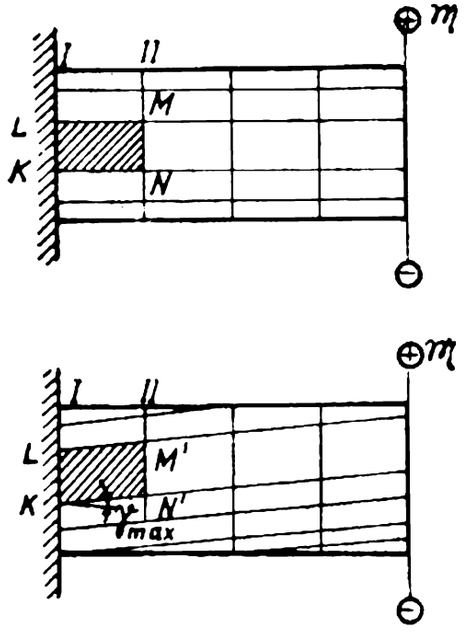


Fig. 5.6

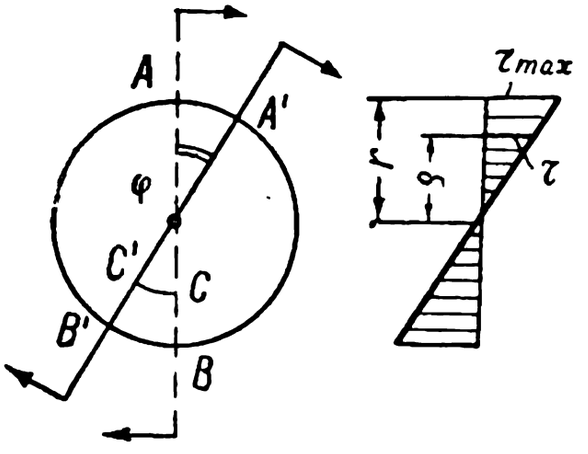


Fig. 5.7

Thus, in torsion, shearing stresses alone act on transverse and longitudinal sections of a bar, i.e., the state of stress at the points of a bar being twisted is pure shear;

(3) the diameter *AB* of the end section (Fig. 5.7) will rotate through a certain angle φ with respect to its original position, remaining a straight line. It is natural to assume that the radii of the other cross sections will also rotate and remain straight. Formulas obtained on this assumption are supported by experimental results. Point *A* will displace along the arc *AA'*, and point *C* along the smaller arc *CC'*.

To establish the law of distribution of shearing stresses over the cross section of a twisted bar consider the deformations of the bar in greater detail (Figs. 5.6 and 5.8). Fig. 5.8 represents the portion of the bar between the sections *I* and *II* on an enlarged scale and shows the side *KN* of the element *KLMN*.

The shear angle for the element $KLMN$ lying on the surface of the bar (Fig. 5.6) is equal to the ratio of the segment NN' to the length of the element dz (Fig. 5.8)

$$\gamma_{max} = \frac{r\,d\varphi}{dz}. \tag{5.1}$$

If we imagine a cylinder of arbitrary radius ρ to be isolated from the portion of the bar under consideration

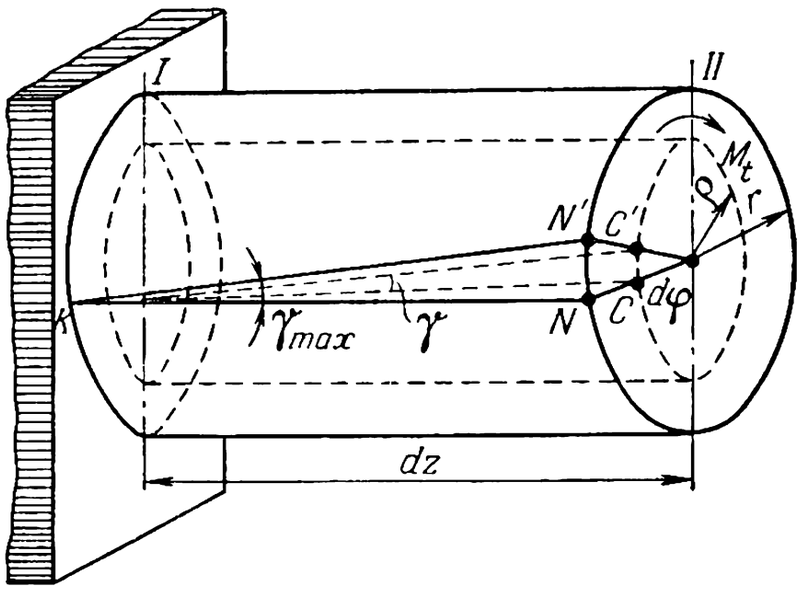


Fig. 5.8

and repeat the same arguments, we obtain the shear angle for an element distant ρ from the axis of the bar

$$\gamma = \frac{\rho\,d\varphi}{dz}. \tag{5.2}$$

On the basis of Hooke's law in shear we have

$$\tau = G\gamma = G\rho\frac{d\varphi}{dz}. \tag{5.3}$$

As can be seen, *the shearing strains and stresses in torsion vary directly with the distance from the centroid of the section.*

The diagram of shearing stresses over the cross section of the bar is represented in the general form at the right-hand side of Fig. 5.7.

At the centroid of a circular section the shearing stress is zero. The maximum shearing stresses occur at the points of the section located at the surface of the bar.

If the law of distribution of shearing stresses is known, their magnitude can readily be determined from the condition of equilibrium.

Consider the equilibrium of the portion of the bar to the right of the section a-a (Fig. 5.9). The algebraic sum of

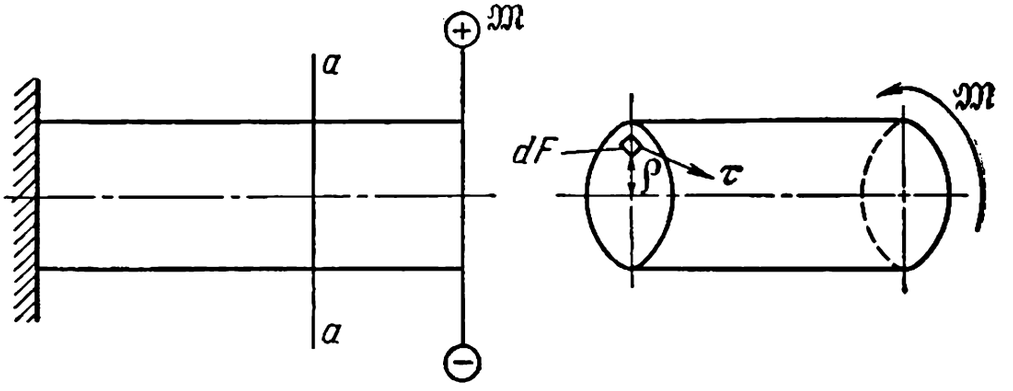


Fig. 5.9

the moments with respect to the axis of the bar acting on the cut portion must be equal to zero

$$\mathfrak{M} - \int_F \rho\tau \, dF = 0, \tag{5.4}$$

where

$\tau\rho \, dF$ = elementary twisting moment of internal forces acting over area dF;

$\int_F \tau\rho \, dF$ = twisting moment at section being considered.

Substituting here the value of the stress from formula (5.3), we obtain

$$\mathfrak{M} - G\frac{d\varphi}{dz}\int_F \rho^2 \, dF = 0. \tag{5.5}$$

Bearing in mind that

$$\int_F \rho^2 \, dF = J_p, \tag{5.6}$$

where J_p is the polar moment of inertia of the cross-sectional area, we obtain

$$\frac{d\varphi}{dz} = \frac{\mathfrak{M}}{GJ_p}. \tag{5.7}$$

Substituting the value of $\frac{d\varphi}{dz}$ in formula (5.3), we have

$$\tau = \frac{\mathfrak{M}}{J_p}\rho. \tag{5.8}$$

If several external moments act on the cut portion of the bar rather than a single external moment $\mathfrak{M}$, then the algebraic sum of these external moments equal in magnitude to the internal twisting moment M_t (see Sec. 34) will enter in formula (5.4) and those which follow.

Thus the final formula for the determination of shearing stresses in torsion is given by

$$\tau = \frac{M_t}{J_p}\rho. \tag{5.9}$$

It follows from this formula that at points equidistant from the centroid of the section the stresses τ are the same.

The maximum stresses at the points near the contour of the section are

$$\tau_{max} = \frac{M_t}{J_p} r = \frac{M_t}{W_p}, \tag{5.10}$$

where

$$W_p = \frac{J_p}{r}. \tag{5.11}$$

The quantity W_p is called *the polar section modulus, or the section modulus in torsion.* It is a geometrical characteristic of a bar of circular cross section in torsion.

For a circular solid section

$$W_p = \frac{J_p}{r} = \frac{\pi d^4}{32d/2} = \frac{\pi d^3}{16} \approx 0.2d^3. \tag{5.12}$$

For a ring section

$$W_p = \frac{2J_p}{D} = \frac{\pi (D^4 - d^4)}{16D} = \frac{\pi D^3}{16}(1-c^4) \approx 0.2D^3(1-c^4), \tag{5.13}$$

where

$$c = \frac{d}{D}.$$

The condition of static strength of the material of a shaft in torsion is

$$\tau_{max} = \frac{M_t}{W_p} \leqslant [\tau]. \tag{5.14}$$

Here $[\tau]$ is the allowable shearing stress.

Under static loads it is assumed that

$$[\tau] = (0.5\text{-}0.6)\,[\sigma_t].$$

With this formula it is not only possible to carry out strength analysis but also to select the shaft diameter or determine the allowable torque when the remaining quantities are known.

Bearing in mind that for a circular solid section $W_p \approx 0.2d^3$ we obtain

$$d = \sqrt[3]{\frac{M_t}{0.2\,[\tau]}} = 1.72\sqrt[3]{\frac{M_t}{[\tau]}}. \qquad (5.15)$$

This formula determines the diameter of a shaft from the condition of strength.

The allowable twisting moment based on the condition of strength is determined by

$$[M_t] = W_p\,[\tau]. \qquad (5.16)$$

The shearing stresses act not only on the transverse sections of the bar but, as follows from the law of conjugate shearing stresses, on the longitudinal sections as well (Fig. 5.10).

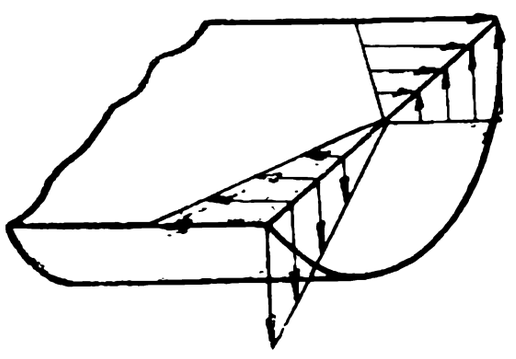


Fig. 5.10

As to oblique sections of the bar both normal and shearing stresses act there. They can be calculated according to the formulas given in Chapter II.

Principal stresses are of the utmost interest. They can be determined according to formula (2.36), substituting in it $\sigma_\alpha = \sigma_\beta = 0$.

In this way we find that $\sigma_1 = \sigma_{max} = \tau$ and $\sigma_3 = \sigma_{min} = -\tau$, the first principal stress being tensile, and the second compressive. According to formula (2.35) we find that the inclinations of the principal areas are $\alpha_0' = 45°$ and $\alpha_0'' = 135°$ (Fig. 5.11*a*).

Experiments show that brittle materials, such as cast iron, fracture in torsion along a plane (strictly speaking, along a helical surface) inclined at 45° to the axis of the shaft (Fig. 5.11*b*), i.e., along those planes where the maximum tensile stresses act.

Consequently, in torsion at all points of a bar except for the points of its axis (at which no stresses occur at all) a biaxial state of stress exists—*pure shear*. The material near the surface of a bar in torsion is more severely stressed than the material nearer the axis of the bar. Thus, the state of stress is non-uniform. If a thin-walled tube is twisted, it may be considered that the same stresses occur

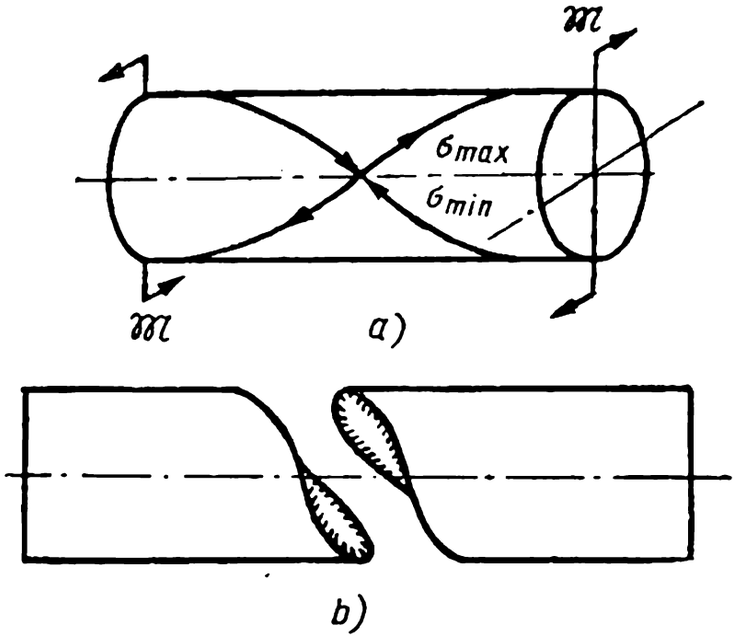


Fig. 5.11

practically at all points of its wall, i.e., in this case the state of stress is uniform. Torsion tests are generally done with such tubes for the purpose of studying pure shear and, in particular, for establishing the magnitude of the yield point in shear τ_y.

36. Relation Between Moment Transmitted by the Shaft, Power and Angular Velocity

In engineering problems the power transmitted by a shaft and also its angular velocity, usually expressed in revolutions per minute, are normally stated. From these data the torque $\mathfrak{M}$ transmitted by the shaft can be calculated. As is known, the power (in horsepower) is given by

$$N = \frac{Pv}{75} = \frac{P2\pi rn}{75 \times 60} \tag{5.17}$$

where v is the speed of a point at the surface of the shaft, n is the number of revolutions of the shaft per minute, P is the circumferential force applied to the shaft.

Since $Pr = \mathfrak{M}$ from formula (5.17) we obtain

$$\mathfrak{M} = 716.2 \frac{N \text{ hp}}{n \text{ rpm}} \text{ kg m}. \tag{5.18}$$

Observing that 1 hp is equal to 0.736 kw we have

$$\mathfrak{M} = \frac{716.2}{0.736} \frac{K}{n} = 973.6 \frac{K}{n} \text{ kg m}, \tag{5.19}$$

where K is the power expressed in kilowatts.

37. Deformations of Circular Bars in Torsion

To calculate deformations of a bar in torsion (angles of twist) we shall use formula (5.7), taking $\mathfrak{M} = M_t$ in accordance with the consideration above

$$d\varphi = \frac{M_t \, dz}{GJ_p}. \tag{5.7}$$

The total angle of twist φ along length z measured from the built-in end will be found by integrating the above expression from 0 to z:

$$\varphi = \int_0^z \frac{M_t \, dz}{GJ_p}. \tag{5.20}$$

If the twisting moment M_t and the quantity GJ_p which is called the torsional rigidity are constant throughout the interval of integration, then

$$\varphi = \frac{M_t z}{GJ_p}. \tag{5.21}$$

As is seen from this formula, the angle of twist increases linearly as the distance from the built-in end (fixed section) grows.

The maximum angle of twist occurs at the end of the bar with $z = l$:

$$\varphi = \frac{M_t l}{GJ_p}. \tag{5.22}$$

This formula is analogous in its structure to the formula for determining deformations in tension-compression.

The angle of twist per unit length is called *the unit angle of twist*. It is given by

$$\theta = \frac{\varphi}{l} = \frac{M_t}{GJ_p}. \tag{5.23}$$

To secure the required rigidity of the shaft it is necessary that the maximum unit angle of twist should not exceed the allowable one:

$$\theta = \frac{M_t}{GJ_p} \leqslant [\theta]. \tag{5.24}$$

This formula expresses *the condition of rigidity* of a circular bar in torsion. In this formula $[\theta]$ is the allowable unit angle of twist in radians per unit length of the shaft.

In most cases the magnitude of the allowable unit angle of twist is given in degrees per metre of length; then in place of formula (5.24) we get

$$\theta = \frac{180}{\pi} \frac{M_t\, 100}{GJ_p} \leqslant [\theta]. \tag{5.25}$$

The magnitude $[\theta]$ is selected in accordance with the function of the shaft and its dimensions. For shafts of medium dimensions it is recommended in the *Handbook of the Mechanical Engineer* to take an allowable angle of twist equal to 0.5° per metre of length.

From condition (5.25) it is possible to determine the shaft diameter based on a given rigidity. Bearing in mind that $J_p \approx 0.1d^4$ we obtain

$$d = \sqrt[4]{\frac{180\, M_t\, 100}{\pi G\, 0.1\, [\theta]}} = B \sqrt[4]{M_t} \tag{5.26}$$

or

$$d = A \sqrt[4]{\frac{N}{n}}, \tag{5.27}$$

where N is the horsepower transmitted by the shaft, n is the angular velocity of the shaft expressed in rpm, the coefficients A and B are determined as a function of $[\theta]$ from Table 5.1 (at $G = 8 \times 10^5$ kg/cm^2), $[\theta]$ is the allowable unit angle of twist in degrees per m; diameter d is obtained in cm.

Table 5.1

Values of Coefficients A and B in Formulas (5.26) and (5.27)

Coefficients	Allowable angle of twist in degrees per metre of shaft					
	0.25	0.5	1.0	1.5	2	2.5
A	12	10	8.5	7.7	7.2	6.8
B	0.73	0.62	0.52	0.47	0.44	0.42

38. Construction of Diagrams for Angles of Twist

To represent visually the manner in which angles of twist vary along the length of a bar, diagrams for angles of twist are sometimes constructed (diagrams for angles of rotation of cross sections).

To illustrate the method of constructing these diagrams let us take the example considered in Section 34

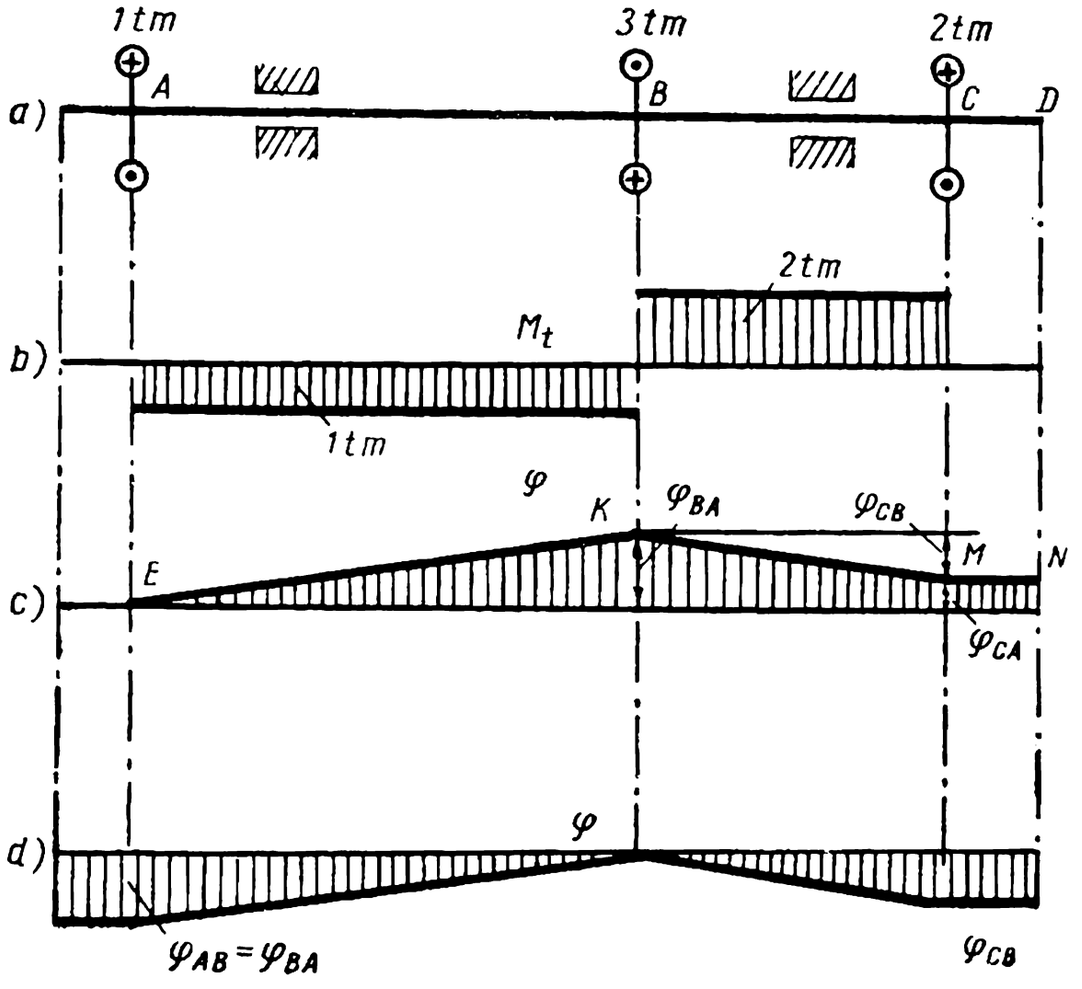


Fig. 5.12

(Fig. 5.12*a*). First construct a twisting moment diagram (Fig. 5.12*b*). Since the shaft is rotating, there are no fixed sections here. But we are interested only in angles of rotation of sections resulting from the deformation of the bar. To construct a diagram of angles of twist any section may conventionally be taken as fixed, for instance section *A*. Determine the rotation of section *B* with respect to section *A*. For that we shall make use of formula (5.22). The angle of twist is determined in terms of the internal twist-

ing moment $M_{AB}=1$ tm acting at cross sections of the bar in portion AB:

$$\varphi_{BA}=\frac{M_{AB}l_{AB}}{GJ_p},$$

where l_{AB} is the length of portion AB.

Let us adopt the following sign rule for angles of twist: angles φ will be considered as positive, if when viewed from left to right the rotation of the section is produced clockwise. In this case φ_{BA} will be positive. To the scale accepted lay off the ordinate φ_{BA} (Fig. 5.12*c*). Join point K thus obtained to point E by a straight line as in portion AB the angles vary according to a linear law. Now calculate the angle of rotation of section C with respect to section B. Taking into account the sign rule adopted for angles of twist we obtain

$$\varphi_{CB}=-\frac{M_{BC}l_{BC}}{GJ_p}.$$

Since section B is not fixed, the angle of rotation of section C with respect to section A will be

$$\varphi_{CA}=\varphi_{CB}+\varphi_{BA}=-\frac{M_{BC}l_{BC}}{GJ_p}+\varphi_{BA}.$$

The angle of twist φ_{CA} may be positive or negative, or in a particular case may equal zero.

Suppose in our case the angle φ_{CA} is positive. Then, laying off this magnitude to the accepted scale upward from the axis of the diagram we obtain point M. Joining this point to point K we obtain the diagram of angles of twist φ in portion BC. In portion CD no twisting occurs ($M_{CD}=0$), which means that all the sections there rotate through the same angle as does section C. Portion MN of the φ diagram is horizontal. The student is recommended to confirm for himself that if section B is taken as a fixed section, the diagram of angles of twist will be as represented in Fig. 5.12*d*.

Example 5.1. Determine the diameter of a steel shaft rotating at an angular velocity of 300 rpm and transmitting 500 hp. The allowable stress $[\tau]=800$ kg/cm^2, the allowable angle of twist $[\theta]=0.5°$ per m, $G=8\times10^5$ kg/cm^2.

Solution. The moment transmitted by the shaft is given by formula (5.18):

$$\mathfrak{M} = 71{,}620 \frac{N}{n} = 71{,}620 \frac{500}{300} = 119 \times 10^3 \text{ kg cm.}$$

The twisting moment at all cross sections of the shaft is the same:

$$M_t = \mathfrak{M} = 119 \times 10^3 \text{ kg cm.}$$

The shaft diameter based on strength is given by the formula

$$d = 1.72 \sqrt{\frac{M_t}{[\tau]}} = 1.72 \sqrt[3]{\frac{119 \times 10^3}{800}} = 9.1 \text{ cm.} \quad (5.15)$$

According to the allowable angle of twist we determine the coefficient $B = 0.62$ from Table 5.1 and according to formula (5.26) the shaft diameter is determined based on the condition of rigidity

$$d = 0.62 \sqrt[4]{119 \times 10^3} = 11.5 \text{ cm.}$$

The diameter of the shaft in this case is determined from the condition of rigidity and must be taken to be $d = 11.5$ cm.

Example 5.2. Find the dimensions of the cross section of a hollow shaft transmitting the moment $\mathfrak{M} = 600$ kgm with the ratio of the diameters $C = \frac{d}{D} = 0.8$ and the allowable stress $[\tau] = 600$ kg/cm². Compare the weight of this hollow shaft with that of a solid shaft of equal strength.

Answer. The dimensions of a hollow shaft: $D = 9.52$ cm, $d = 7.62$ cm. The cross-sectional area $F_h = 25.9$ cm². The diameter of a solid shaft $d_1 = 8$ cm. The cross-sectional area $F_s = 50.2$ cm². The weight of the hollow shaft is 51 per cent of the weight of the solid shaft.

39. Potential Energy in Torsion

In torsion external moments do work by reason of the rotation of sections to which they are applied. This work is consumed in the storage of potential energy of strain numerically equal to the work done by internal forces.

Just as with tension, it can be proved that the work done by a statically applied external torque is equal to one-half the product of the final value of the torque and the magnitude of the final angle of twist:

$$A = \frac{1}{2} \mathfrak{M}_1 \varphi_1, \tag{5.28}$$

The work done by internal forces, negative in sign, and the potential energy numerically equal to it are calculated in torsion just as the work done by internal forces was calculated in tension (compression).

The elementary work done by internal forces is

$$dU = -\frac{1}{2} M_t \, d\varphi,$$

where M_t is the internal twisting moment, $d\varphi$ is the angle of twist for the element of length dz. But according to formula (5.7)

$$d\varphi = \frac{M_t \, dz}{GJ_p}.$$

Therefore

$$dU = -\frac{1}{2} \frac{M_t^2 \, dz}{GJ_p}.$$

The total work done by internal forces will be obtained by integrating the expression for the elementary work over the entire length l of the bar:

$$U = -\frac{1}{2} \int_0^l \frac{M_t^2 \, dz}{GJ_p}.$$

For calculating the potential energy equal in magnitude to the work done by internal forces and opposite in sign we have

$$\Pi = -U = \frac{1}{2} \int_0^l \frac{M_t^2 \, dz}{GJ_p}. \tag{5.29}$$

If the magnitude of the twisting moment and the rigidity of the bar do not vary along the length, then

$$\Pi = \frac{1}{2} \frac{M_t^2 l}{GJ_p}. \tag{5.30}$$

This formula is completely analogous in structure to the formula $\Pi = \frac{1}{2}\frac{N^2 l}{EF}$ for calculating the potential energy in tension (compression).

40. Basic Results of Theory of Torsion of Non-Circular Bars

In such bars the cross sections do not remain plane during twist but *warp* (*distort*) (Fig. 5.13*a*).

If such warping is not restrained, then no normal stresses occur at cross sections. Such torsion is called *pure* or *free*.

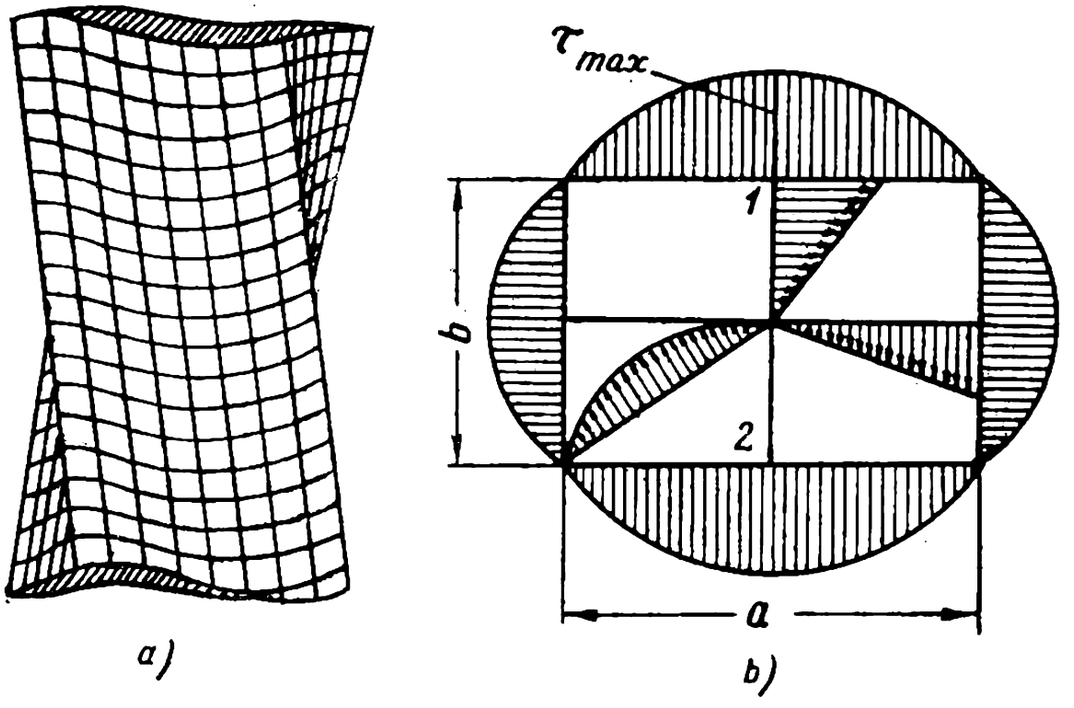


Fig. 5.13

Free torsion is possible only if the cross section of the bar and the magnitude of the twisting moment are constant, and the ends of the bar are not built-in.

Determination of shearing stresses for non-circular bars is a somewhat complicated problem which is solved by the methods of the theory of elasticity. Let us examine the basic results for bars of rectangular cross section with $a > b$ (Fig. 5.13*b*).

The maximum shearing stresses occur at points *1* and *2*, i.e., in the middle of the longer sides, and are given by

$$\tau_{max} = \frac{M_t}{\alpha a b^2} = \frac{M_t}{W_t}. \qquad (5.31)$$

The angle of twist is

$$\varphi = \frac{M_t l}{G\beta ab^3} = \frac{M_t l}{GJ_t}. \tag{5.32}$$

The quantities $W_t = \alpha ab^2$ and $J_t = \beta ab^3$ are respectively geometrical characteristics of the strength and rigidity of a bar of rectangular cross section in torsion.

Here α and β are factors depending upon the ratio of the sides. Their values are given in Table 5.2.

Table 5.2

Values of the Factors α and β for the Twist of Bars of Rectangular Cross Section

$a:b$	1	2	3	4	5	10	20	∞
α	0.21	0.25	0.27	0.28	0.29	0.31	0.32	0.33
β	0.14	0.23	0.26	0.28	0.29	0.31	0.32	0.33

As is seen from the table, for narrow rectangular cross sections with the ratio $\frac{a}{b} \geqslant 10$ we may take $\alpha = \beta \approx 1/3$.

The distribution of shearing stresses along the perimeter of the cross section of the bar, along its axes and the diagonals of the cross section is seen from Fig. 5.13*b*. At the corners $\tau = 0$.

For other profile sections (elliptic, circular with a key-groove, etc.) design formulas are given in handbooks.

For cross sections composed of long thin rectangles $\left(\frac{a}{b} \geqslant 10\right)$ the geometrical characteristic J_t can be calculated according to the formula

$$J_t = \eta \frac{1}{3} \Sigma ab^3. \tag{5.33}$$

where b is the length of the shorter side of the rectangle, a is the length of the longer side of the rectangle, and η

is a factor depending upon the shape of the cross section.

Shape of section	*Value of the factor* η
angle	1.00
I-section	1.20
T-section	1.15
channel	1.12

The maximum shearing stress occurs in the rectangle with the greatest width and is given by

$$\tau_{max} = \frac{M_t b_{max}}{J_t} \tag{5.34}$$

where b_{max} is the greatest value of the shorter side of the rectangle.

The angle of twist is determined according to the formula

$$\varphi = \frac{M_t l}{G J_t}. \tag{5.35}$$

For rolled sections the value of J_t is given in special tables. It should be noted that the magnitude of J_t for such sections (thin-walled open sections) is very small as compared with the magnitude of J_t for solid circular bars having the same cross-sectional area, to say nothing of those with a ring section.

For instance, the value of J_t for I-beam No. 40a (GOST* 10016-39) is 68.75 cm^4 which is $^1/_{11}$ th of the polar moment of inertia of a solid circular section of the same area. Bars having such sections should not therefore be used in torsion.

In *non-free (constrained)* torsion when cross sections are prevented from warping the foregoing formulas are not applicable. The general theory of constrained torsion of thin-walled bars of open section was developed by V. Z. Vlasov (U.S.S.R.). He showed that in constrained torsion, besides the shearing stresses of pure torsion as calculated according to the foregoing formulas, considerable additional shearing and normal stresses arise at the cross section. The theory of constrained torsion of thin-walled bars is not dealt with in the elementary course on strength of materials.

* GOST (ГОСТ) — U.S.S.R. State Standard.

41. Torsion of Thin-Walled Bars of Closed Section

Thin-walled bars of closed section are far more rigid and their use is therefore more advisable in torsion.

Consider a cylindrical bar whose cross section is represented in Fig. 5.14. The wall thickness δ will be considered to vary smoothly along the contour so that stress concentration can be neglected.

Since the wall thickness is small, it may be assumed that shearing stresses developed in torsion will be uniformly distributed throughout the thickness.

Let us introduce further the following hypothesis, which is supported by experiments.

The product of the shearing stress at any point of the wall by its thickness at that place is a constant quantity for all points of the axial line of the contour of the cross section. This condition is written mathematically as follows:

$$\tau\delta = C.$$

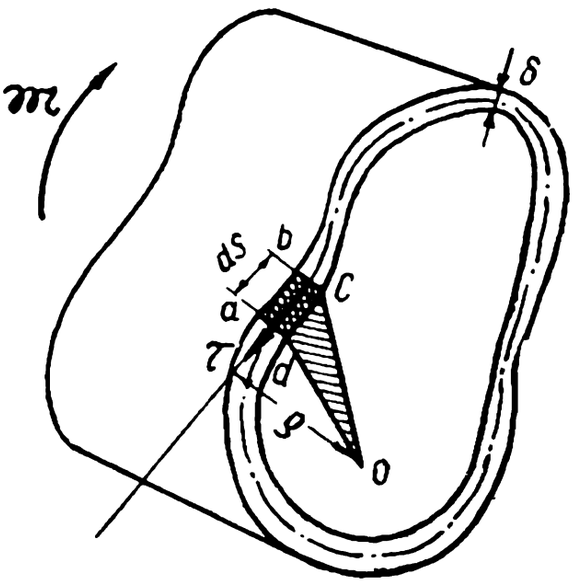


Fig. 5.14

The magnitude of the shearing stress can now be related to the twisting moment developed in a section. For that consider the equilibrium of a portion of the bar (see Fig. 5.14).

The force acting on an elementary area δds is $\tau\delta ds$ and the twisting moment of this elementary force with respect to any point O lying in the plane of the section is $\tau\delta ds\rho$, where ρ is the arm of the force with respect to the point O.

For the cut portion of the bar to be in equilibrium the sum of the moments about the axis parallel to the generatrix of the bar and passing through point O must be equal to zero:

$$\mathfrak{M} - \int_s \tau\delta\rho\, ds = 0,$$

where integration is extended over the entire length s of the contour. But the product ρds is equal to twice the area of the triangle Oab:

$$\rho\, ds = 2\, d\omega.$$

Therefore

$$\mathfrak{M} - \int_{\omega} \tau\delta\, 2d\omega = 0.$$

The product $\tau\delta$, as a constant quantity, may be taken out of the integral sign. The integral will become $\int_{\omega} d\omega$, which represents the area of the solid cross section bounded by the centre line of the wall thickness. Then

$$\mathfrak{M} - \tau\delta 2\omega = 0. \tag{5.36}$$

In the case of several external moments this formula will include the sum of external moments acting on the cut portion which represents in magnitude the internal twisting moment in the section M_t. Taking this into account we obtain from formula (5.36)

$$\tau = \frac{M_t}{\delta 2\omega}. \tag{5.37}$$

The maximum stress will occur where the wall is thinnest:

$$\tau_{max} = \frac{M_t}{\delta_{min} 2\omega}. \tag{5.38}$$

The angle of twist φ for a bar of length l will be determined from the condition that the work done by the external torque must be equal to the work done by the internal forces. The work done by an external statically applied moment $\mathfrak{M}$ on an angular displacement φ is

$$A = \frac{\mathfrak{M}\varphi}{2}.$$

Now calculate the potential energy of strain numerically equal to the work done by the internal forces. The unit potential energy when the bar is subjected to shearing stresses is given by formula (3.15)

$$u = \frac{\tau^2}{2G}.$$

The potential energy for an element of the wall with the volume $\delta l ds$ will be

$$d\Pi = \frac{\tau^2}{2G}\delta l\, ds,$$

where l is the length of the bar. The total potential energy for the entire bar will be

$$\Pi = \frac{l}{2G}\int_0^s \tau^2 \delta\, ds.$$

Integration is extended over the length s of the contour of the cross section.

Replacing τ by its value from formula (5.36) we obtain

$$\Pi = \frac{l}{2G}\int_0^s \frac{\mathfrak{M}^2}{\delta^2 4\omega^2}\,\delta\, ds.$$

Taking the constants out of the integral sign we have

$$\Pi = \frac{\mathfrak{M}^2 l}{8G\omega^2}\int_0^s \frac{ds}{\delta}.$$

Since the potential energy Π is numerically equal to the work A done by the external moment we obtain

$$\varphi = \frac{\mathfrak{M} l}{4G\omega^2}\int_0^s \frac{ds}{\delta}. \qquad (5.39)$$

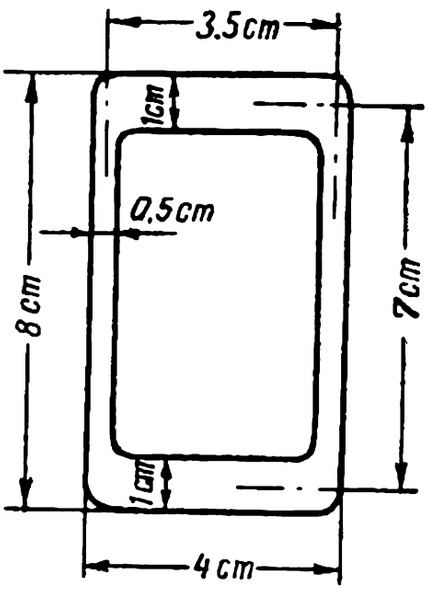


Fig. 5.15

Example 5.3. Determine the maximum stress and the angle of twist for a hollow bar (Fig. 5.15), if $\mathfrak{M} = 150$ kgm, $G = 8\times 10^5$ kg/cm^2.

Solution. According to formula (5.38) we find ($M_t = \mathfrak{M}$)

$$\tau_{max} = \frac{150\times 10^2}{2\times 3.5\times 7\times 0.5} = 612 \text{ kg/cm}^2.$$

According to formula (5.39) we determine the angle of twist in a length of 1 m

$$\varphi = \frac{150\times 10^2\times 100}{4\times 3.5^2\times 7^2\times 8\times 10^5}\left(\frac{2\times 3.5}{1}+\frac{2\cdot 7}{0.5}\right) = 0.027 \text{ radians}.$$

Example 5.4. Determine the maximum stress and the angle of twist for the same bar if the section is open (i.e., if the contour is cut in one place).

Solution. The stress is found according to formula (5.34). The value of η is taken as for a channel $\eta = 1.15$:

$$\tau_{max} = \frac{M_t b_{max}}{\eta \frac{1}{3} \Sigma a b^3} = \frac{15 \times 10^3 \times 1}{1.15 \times \frac{1}{3} (2 \times 3.5 \times 1^3 + 2 \times 7 \times 0.5^3)} =$$

$$= \frac{15 \times 10^3}{3.36} = 4{,}460 \text{ kg/cm}^2.$$

Note that this result is true only for a bar made of alloy steel the proportional limit of which in pure shear τ_p is not less than the ascertained magnitude of τ_{max} since all the formulas given in this chapter hold only when Hooke's law is valid.

The angle of twist is found according to formula (5.35):

$$\varphi = \frac{15 \times 10^3 \times 100}{8 \times 10^5 \times 3.36} = 0.56 \text{ radians}.$$

A comparison of the results of the two examples considered confirms the advantages of bars of closed section over bars of open section when subjected to torsion.

42. Statically Indeterminate Problems

In torsion, just as in tension, problems are encountered that cannot be solved by means of equilibrium equations alone. In such problems the number of unknowns exceeds the number of equations of equilibrium.

The procedure for solving such problems is the same as for solving statically indeterminate problems in tension (compression).

By way of example let us consider a bar with two built-in ends (Fig. 5.16*a*). Such a bar is statically indeterminate since statics gives only one equation of equilibrium for determining two reactive moments arising at the built-in ends.

Remove one fixed end, replacing its action by an unknown moment X (Fig. 5.16*b*). The additional equation (called the equation of deformations or the equation of displacements) will be obtained from the condition that the angle of rotation of the section at the removed fixed

end equal to the angle of twist of the bar due to the moments $\mathfrak{M}_1$ and X is zero: $\varphi_B = 0$.

In the statically determinate system thus obtained the rotation of section B will be produced by the external moment and moment X. The angle of rotation of section B under the action of moment X is

$$\varphi'_B = \frac{Xa}{GJ'_p} + \frac{Xb}{GJ''_p},$$

where

$$J'_p \approx 0.1d_1^4, \quad J''_p \approx 0.1d_1^4.$$

The angle of rotation of section B under the action of moment $\mathfrak{M}_1$ is

$$\varphi''_B = -\frac{\mathfrak{M}_1 a}{GJ'_p}.$$

Substituting these values in the equation of deformations we obtain

$$\frac{Xa}{GJ'_p} + \frac{Xb}{GJ''_p} - \frac{\mathfrak{M}_1 a}{GJ'_p} = 0.$$

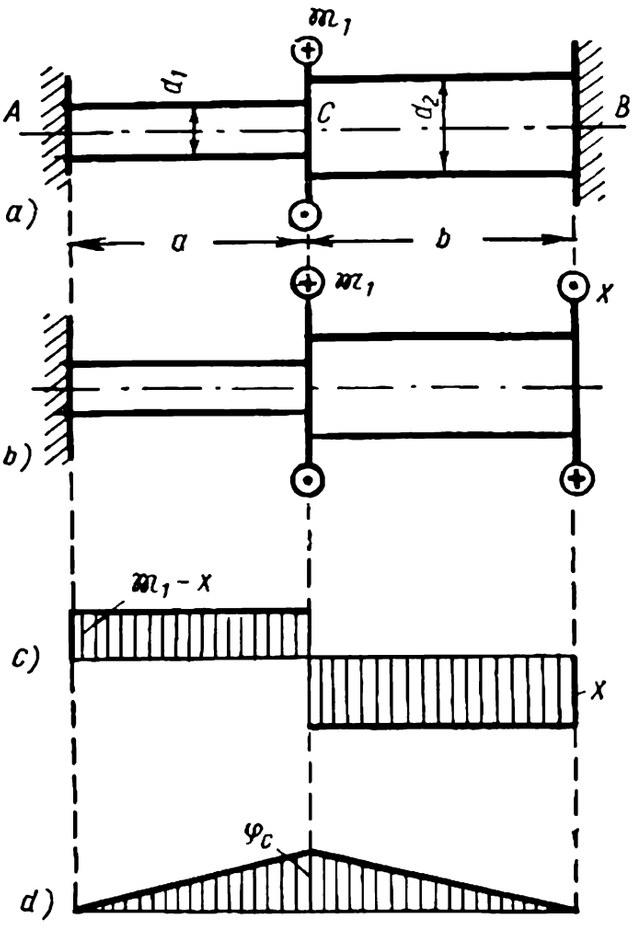


Fig. 5.16

Hence X can be determined. After this we can find the twisting moment in any section and construct the diagrams of M_t and angles of twist. To construct the φ diagram it is sufficient to calculate the angle of rotation of section C. It is given by

$$\varphi_C = \frac{Xb}{GJ''_p} = \frac{(\mathfrak{M}_1 - X)a}{GJ'_p}.$$

The angles of rotation of sections A and B are zero and as the angle of rotation of a section depends linearly on the distance, the points obtained may be joined by straight lines. The M_t and φ diagrams are represented in Fig. 5.16 *c* and *d*.

Example 5.5. A thin-walled tube made of a material having a modulus G_i is inserted into another one having

a modulus G_o. One end of the resulting construction is fixed, while the other is acted upon by an external moment $\mathfrak{M}$ applied to both tubes (Fig. 5.17). Determine the twisting moments developed in cross sections of the tubes.

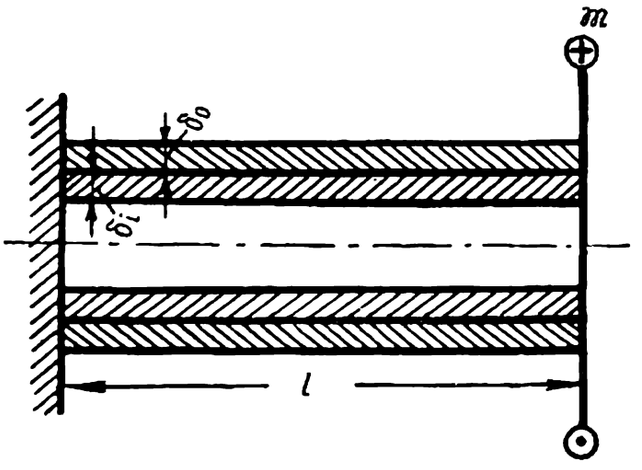


Fig. 5.17

Solution. There are two unknown twisting moments: $M_{t.i}$ in the inside tube and M_{to} in the outside tube.

There is only one equation of equilibrium:

$$M_{t.i} + M_{t.o} = \mathfrak{M} \quad \text{(I)}$$

The problem is once statically indeterminate. By equating the angles of rotation of the sections at the right end of the tubes (equal to the total angles of twist for the tubes) we derive the equation of deformations:

$$\varphi_i = \varphi_o$$

or

$$\frac{M_{t.i}l}{G_i J_{p.i}} = \frac{M_{t.o}l}{G_o J_{p.o}}. \quad \text{(II)}$$

Here J_{pi} denotes the polar moment of inertia of the section for the inside tube and J_{po} for the outside tube. They are determined as for hollow sections according to the formulas given in Section 29. When the wall thickness is small the angles of twist can be calculated by using formula (5.39) which for the constant thickness δ becomes

$$\varphi = \frac{M_t l s}{4G\omega^2\delta} = \frac{4M_t l}{G\pi d^3\delta},$$

where d is the mean diameter of the tube

$$d = \frac{d_o + d_i}{2},$$

$s = \pi d$ is the length of the middle circumference of the tube section.

From the two equations I and II the twisting moments in cross sections of the tubes are determined, after which

stresses are obtained by use of formula (5.37). When the wall thickness is considerable the formulas given in Section 35 should be employed for calculating stresses.

43. Stress Concentration

An abrupt change in the contour of a transverse or longitudinal section of a shaft gives rise to *stress concentration.* The effect of stress concentration is taken into account

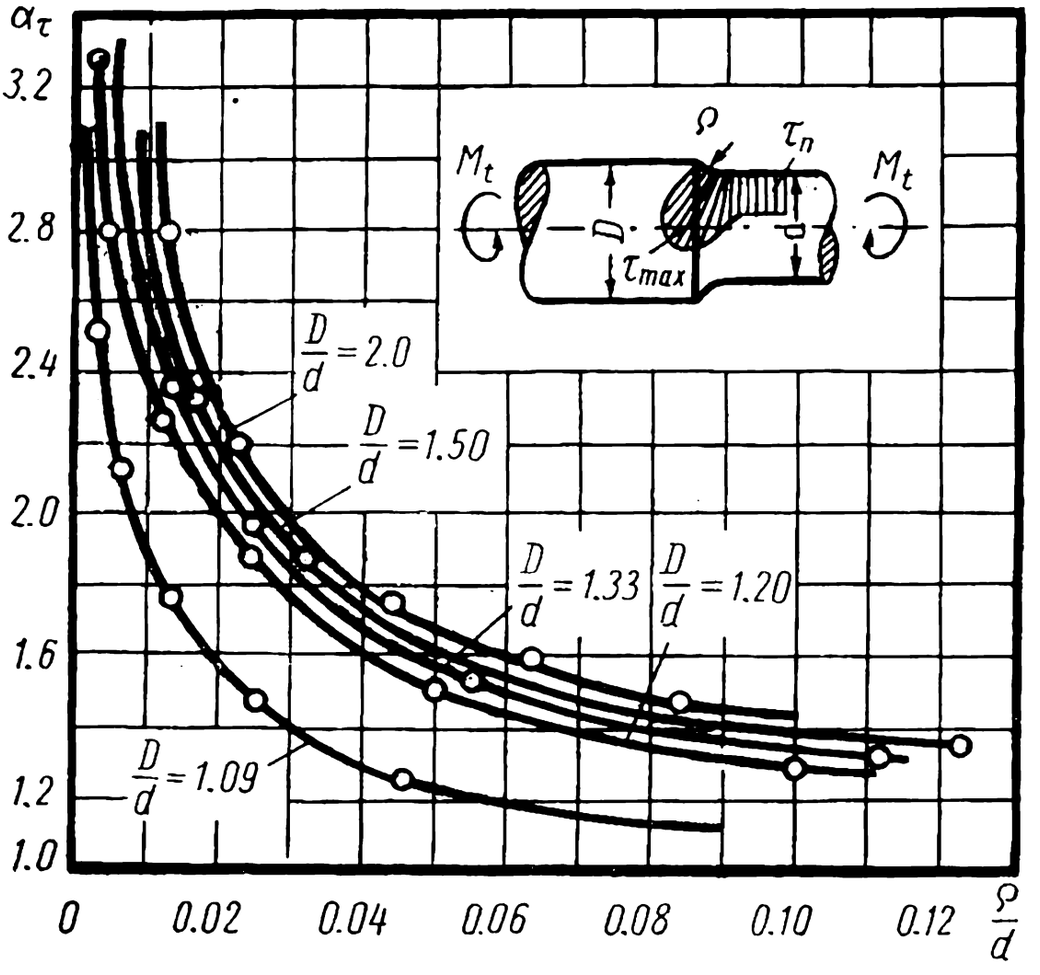


Fig. 5.18

by the stress concentration factor which is determined either theoretically, by the methods of the theory of elasticity, or experimentally.

Fig. 5.18 gives a graph for determining the magnitude of the theoretical stress concentration factor α_τ for a torsional shaft whose parts are joined along a circular fillet of radius ρ. As can be seen, for sharp discontinuities, i.e., with small values of ρ/d, the magnitude of α_τ is greatly increased.

When the stress concentration factor is known the maximum shearing stress for a circular bar is given by

$$\tau_{max} = \alpha_\tau \tau_n = \alpha_\tau \frac{M_t}{W_p},$$

where $\tau_n = \frac{M_t}{W_p}$ is the nominal stress as calculated across the least section.

The influence of stress concentration is to be taken into consideration:

(a) for low-ductile materials and materials that are apt to fracture in a brittle manner when subjected to static loads;

(b) for all materials subjected to alternating loads (fatigue analysis).

To reduce stress concentration abrupt changes in the contour of a section should be avoided by employing transition curves of the largest possible radius at points where there is a stepwise variation in the diameter of a bar.

44. Rational Shapes of Sections in Torsion

Of two cross sections having the same polar section modulus (or having the same W_t in the case of a non-circular cross section) and consequently the same allowable twisting moment, the section with the smaller cross-sectional area will be rational, i.e., will entail the least consumption of material. As the ratio $\frac{W_p}{F}\left(\text{or } \frac{W_t}{F}\right)$ is a dimensional quantity, in order to compare various cross sections it is convenient to employ the dimensionless quantity

$$w_p = \frac{W_p}{\sqrt{F^3}}$$

$\left(\text{for a non-circular section: } w_t = \frac{W_t}{\sqrt{F^3}}\right)$, which may be termed the unit section modulus in torsion.

In Table 5.3 are given the values of w_t for some cross sections.

As can be seen, channels, I-sections and narrow rectangular sections are least economical in torsion, and circular ring sections are the most economical, particularly when

the wall thickness is small, i.e., at the values of $c = \frac{d}{D}$ close to unity.

Table 5.3

Unit Section Moduli in Torsion for Bars of Various Cross Sections

Type of section	w_t
Channel	0.04-0.05
I-section	0.05-0.07
Rectangular section with $\frac{a}{b} = 10$	0.1
Rectangular section with $\frac{a}{b} = 2$	0.18
Square	0.21
Solid circular section	0.28
Circular ring with $c = \frac{d}{D} = 0.5$	0.37
Circular ring with $c = 0.9$	1.16

Let us compare the weight of bars of tubular section G_T with the weight of bars of solid section G_S having equal strength for different values of $c = \frac{d}{D}$. By equating the polar section moduli of solid and ring sections we have

$$\frac{\pi d_1^3}{16} = \frac{\pi D^3}{16}(1 - c^4).$$

To secure equal strength the following condition must be satisfied

$$D = \frac{d_1}{\sqrt[3]{1 - c^4}}.$$

The ratio of the cross-sectional areas (and, consequently, of the weights) will be

$$\frac{G_T}{G_S} = \frac{\pi}{4}(D^2 - d^2) : \frac{\pi d_1^2}{4} = \frac{D^2(1 - c^2)}{d_1^2}.$$

Substituting here the value of D obtained from the condition of equal strength we find

$$G_T : G_S = \frac{1 - c^2}{\sqrt[3]{(1 - c^4)^2}}.$$

In Table 5.4 are given the values of $G_T : G_S$ as calculated according to this formula for different values of $c = \frac{d}{D}$.

Table 5.4

The Ratio of Weights of Bars of Solid Circular and Ring Sections with Equal Strength in Torsion

c	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\frac{G_T}{G_S}$	1	0.99	0.96	0.92	0.85	0.79	0.70	0.61	0.51	0.39

It is seen from this table that the use of tubular thin-walled bars means a great saving of metal.

In choosing sections based on rigidity the following non-dimensional quantity may be used as a criterion of an economical section:

$$j_p = \frac{J_p}{F^2}$$

$$\left(\text{or } j_t = \frac{J_t}{F^2} \text{ for non-circular sections}\right)$$

which may be termed the unit polar moment of inertia or the unit geometrical characteristic of torsional rigidity.

Table 5.5 gives the values of j_t for some of the most commonly used sections.

Table 5.5

Type of section	j_t
Channel	0.010-0.011
I-section	0.009-0.015
Rectangular section with $\frac{a}{b} = 10$	0.031
Rectangular section with $\frac{a}{b} = 2$	0.115
Square	0.14
Solid circular section	0.16
Circular ring with $c = \frac{d}{D} = 0.5$	0.264
Circular ring with $c = 0.9$	1.52

As can be observed, in a design based on rigidity the advantages of thin-walled ring sections over other types are still more pronounced. A comparison of weights of bars of circular ring and solid sections with equal rigidity is given in Table 5.6, in which G_T denotes the weight of a bar of circular tubular section and G_S denotes the weight of a bar of solid circular section.

Table 5.6

c	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\frac{G_T}{G_S}$	1	0.99	0.96	0.92	0.85	0.78	0.69	0.58	0.46	0.32

Comparing this table with Table 5.4 we see that in a design based on rigidity the use of thin-walled tubular bars makes possible a further saving of material.

CHAPTER VI

BENDING. DETERMINATION OF STRESSES

45. Basic Concepts in Bending

It is often the case that a bar is subjected to a transverse load or external couples whose plane of action passes through the axis of the bar (Fig. 6.1).

At cross sections of the bar there arise bending moments, i.e., internal moments whose plane of action is perpendicular to the plane of the cross section of the bar.

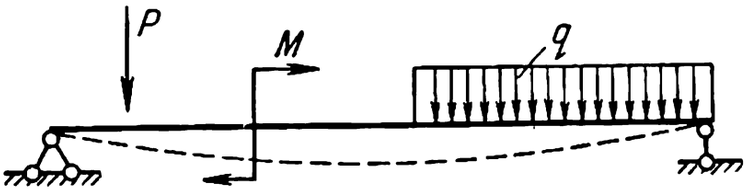


Fig. 6.1

The axis of the bar bends under the action of such a load.

This kind of loading is called bending. Bars working primarily in bending are usually called beams. Bending is described as pure if the bending moment is the only internal force at the cross section of the bar.

It is more often the case, however, that there arise also shearing forces as well as bending moments at cross sections of the bar. Such bending is described as *transverse.*

If the plane of action of the bending moment passes through one of the principal centroidal axes of inertia of the cross section of the bar, bending is described as *plane.* Otherwise bending is referred to as *oblique.*

In plane bending the axis of the beam continues to remain in the plane of action of external forces, the plane of loading, after deformation. In oblique bending the plane of deformation does not coincide with the plane of loading.

Plane bending of thin-walled bars with a load acting in a plane passing through a principal axis of inertia which is not an axis of symmetry for the cross section of the bar is complicated by torsion (see Sec. 54).

To study deformation in bending we shall begin with a case of pure plane bending; later we shall consider the more general type of bending, transverse bending. Oblique bending is connected with complex resistance of bars and will be discussed in Chapter IX.

46. Types of Supports of Beams

Supports of beams which are considered as plane systems can be of three basic types:

1. *A movable hinged support* (Fig. 6.2*a*). Such a support does not prevent the end of the beam from rotating

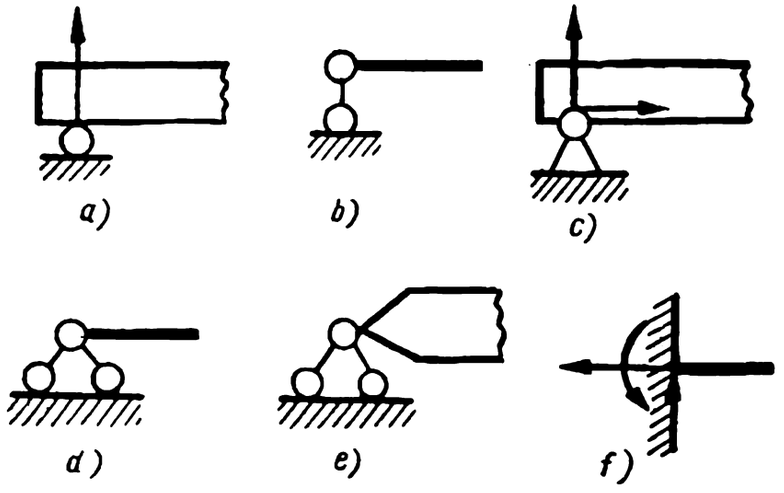


Fig. 6.2

and its displacement along the plane of rolling. Only one reaction can arise in this support—perpendicular to the plane of rolling and passing through the centre of the roller.

A movable support is represented schematically as one rod with hinges at its ends (Fig. 6.2*b*).

Movable supports permit a beam to change its length freely with variations in temperature and thereby eliminate a possible occurrence of thermal stresses.

2. *An immovable hinged support* (Fig. 6.2*c*). Such a support permits the end of the beam to rotate but prevents its translational displacement in any direction. The reaction

arising in this support can be resolved into two components—the horizontal and the vertical. A schematic representation of a hinged immovable support is shown in Fig. 6.2 *d* and *e*; the supporting hinge must be located at the level of the axis of the beam. If this condition is not fulfilled, an appreciable error may be introduced in design due to the elongation or contraction of fibres of the beam which in bending is restrained by an immovable hinge support.

3. *Rigid fixing, or clamping* (Fig. 6.2*f*). Such fixing prevents both linear and angular displacements of the section over the support. This support can produce in general a reaction which is usually resolved into two components, the vertical and the horizontal, and a clamping moment (reactive moment).

A beam with one built-in end is called a cantilever beam, or simply a *cantilever.*

If the reactions of the supports can be found from the equations of statics alone, beams are called *statically determinate.* If, however, the number of unknown reactions is greater than the number of equations of statics corresponding to a given problem, beams are called *statically indeterminate.* For determining the reactions in such beams it is necessary to derive additional equations—equations of deformations.

47. Determination of Reactions of Supports

Consider several examples.

Example 6.1. Determine the reactions of the support for the cantilever beam (Fig. 6.3).

Solution. The reaction arising in the built-in end can be represented as two forces A_z and A_y directed as shown in the drawing, and a reactive moment M_A.

Derive the equations of equilibrium of the beam.

1. $\Sigma Z=0$. We obtain $A_z=0$. In the absence of a horizontal load the horizontal component of the reaction is zero.

2. $\Sigma Y=0$. The uniformly distributed load q is replaced by the resultant qa_3 applied at the middle of the portion a_3:

$$A_y - P_1 - qa_3 = 0,$$

whence

$$A_y = P_1 + qa_3.$$

The vertical component of the reaction in a cantilever beam is equal to the sum of the forces applied to the beam.

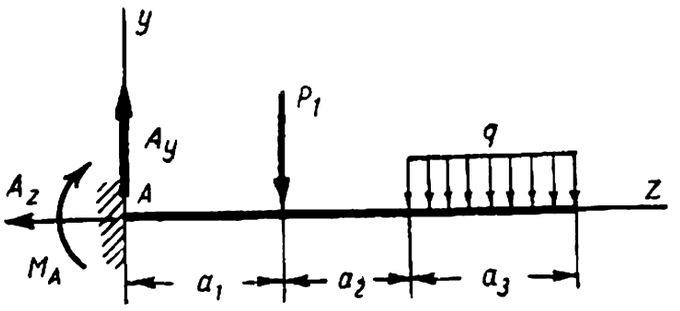


Fig. 6.3

3. Derive the third equation of equilibrium. Equate to zero the sum of the moments of all forces with respect to any point, for instance, with respect to point A:

$$\Sigma M_A = 0, \quad -M_A - P_1 a_1 - qa_3\left(a_1 + a_2 + \frac{a_3}{2}\right) = 0,$$

whence

$$M_A = -P_1 a_1 - qa_3\left(a_1 + a_2 + \frac{a_3}{2}\right).$$

The negative sign indicates that the assumed direction of the reactive moment should be reversed. Thus, the reac-

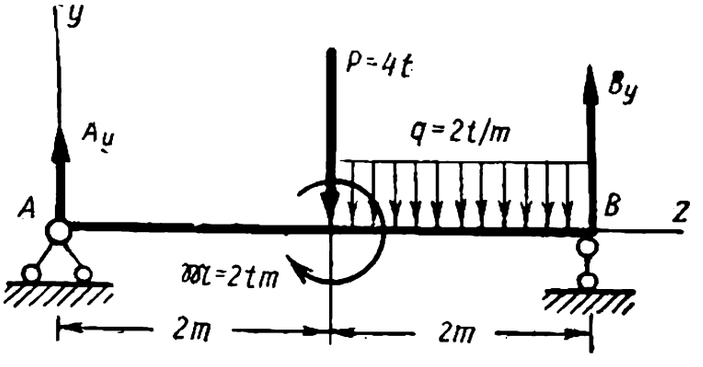


Fig. 6.4

tive moment at the built-in end is equal to the sum of the moments of external forces with respect to the built-in end.

Example 6.2. Find the reactions of the supports for a beam with both ends supported (Fig. 6.4). Such beams are usually called simple beams.

Solution. Since there is no horizontal load, $A_z=0$:

(1) $\Sigma M_B=0$, $-A_y\times 4+4\times 2+4\times 1-2=0$, $A_y=2.5$t,

(2) $\Sigma M_A=0$, $B_y\times 4-4\times 3-4\times 2-2=0$, $B_y=5.5$t.

In place of the second equation the condition $\Sigma Y=0$ could be used. Here it should be employed to check the solution.

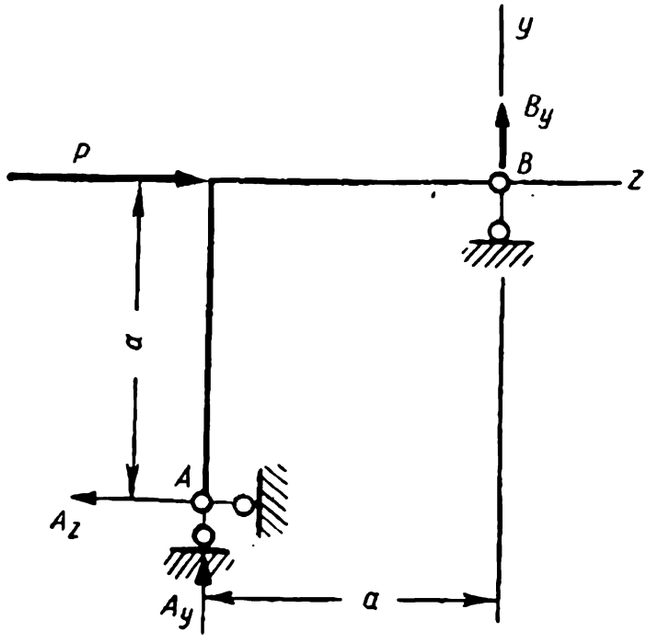


Fig. 6.5

Example 6.3. Determine the reactions of the supports for a beam with a broken contour (Fig. 6.5).

Solution.

(1) $\Sigma M_A=0$, $-Pa+B_ya=0$, $B_y=P$,

(2) $\Sigma Z=0$, $-A_z+P=0$, $A_z=P$,

(3) $\Sigma Y=0$, $A_y+B_y=0$, $A_y=-B_y=-P$,

i.e., the reaction A_y acts downwards and not upwards. For checking the solution use can be made, for instance, of the equation $\Sigma M_B=0$.

48. Determination of Internal Forces

As already noted, in plane transverse bending two internal forces (internal force factors) are set up at cross sections of a beam—bending moment M_b and shearing force Q. To determine them we shall apply the method of

sections. Imagine the beam cut at a certain place, for instance, at a distance z from the left support (Fig. 6.6*a*). Remove one of the portions of the beam, say, the right one, and consider the equilibrium of the left portion.

The interaction of the portions of the beam will be replaced by internal forces: the bending moment M_b and the shearing force Q (Fig. 6.6*b*).

The magnitude of M_b and Q will be determined by the use of two equations of equilibrium:

(1) $\Sigma Y = 0,\ A - P_1 + Q = 0,$

$$Q = P_1 - A,\ Q = \Sigma (P_i)_y,$$

(2) $\Sigma M_o = 0,$

$$Az - P_1 (z - a_1) - M_b = 0,$$

$$M_b = Az - P_1 (z - a_1),$$

$$M_b = \Sigma m_o (P_i).$$

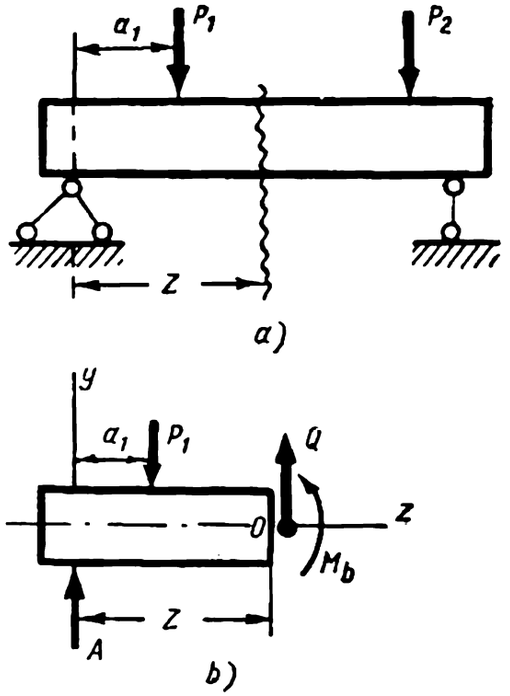


Fig. 6.6

Thus

(1) *shearing force Q at the cross section of the beam is numerically equal to the algebraic sum of the projections on the plane of the section of all the external forces acting to one side of the section;*

(2) *the bending moment at the cross section of the beam is numerically equal to the algebraic sum of the moments, taken with respect to the centroid of the section, of the external forces acting to one side of this section.*

49. Sign Rule for Bending Moments and Shearing Forces

The shearing force at a cross section of a beam, for instance at the section *mn* (Fig. 6.7*a*), is considered to be positive if the resultant of the external forces to the left of the section acts upwards, and to the right of the section downwards; otherwise it is considered to be negative (Fig. 6.7*b*).

The bending moment at a cross section of a beam, for instance at the section *mn* (Fig. 6.8*a*), is considered to be positive if the resultant moment of the external forces to

the left of the section is clockwise, and to the right of the section counter-clockwise; otherwise it is considered to be negative (Fig. 6.8*b*). The moments shown in Fig. 6.8*a* produce bending of the beam concave upward, and the moments shown in Fig. 6.8*b* produce bending of the beam concave downward. This can be readily verified by bending a thin ruler.

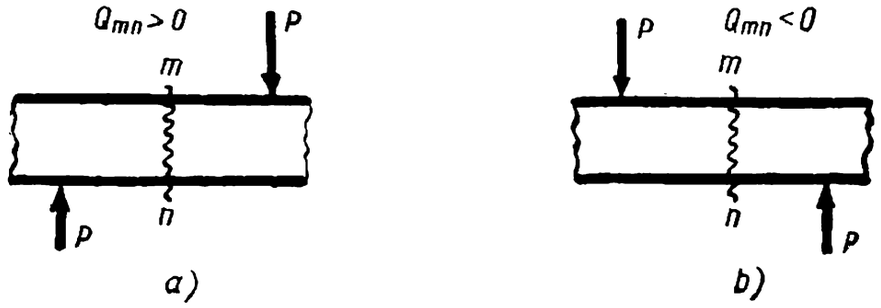


Fig. 6.7

An alternative and easier to memorise sign rule for the bending moment is as follows. *The bending moment is considered to be positive if it produces bending of the beam concave upward at the section considered.* It is evident that the fibres of the beam located in the concave part

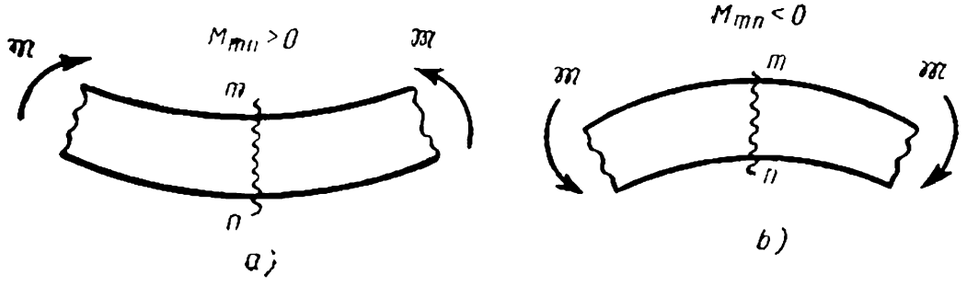


Fig. 6.8

undergo compression and those in the convex part extension. Thus, if we decide to plot positive ordinates of the M_b diagram above the base line of the diagram (which is parallel to the axis of the beam), the diagram is found to be *on the side of the compressed fibres of the beam.*

50. Relation Between Bending Moment, Shearing Force and Intensity of Distributed Load

A relation can easily be established between the bending moment, shearing force and intensity of distributed load.

Consider a beam subjected to an arbitrary load (Fig. 6.9). Determine the shearing force at a section at a distance z from the left support. Projecting the forces acting to the left of the section on the vertical direction we obtain *

$$Q_z = A - P_1 + qz. \tag{a}$$

Similarly, calculate the shearing force at an adjacent section at a distance $z+dz$ from the left support:

$$Q_z + dQ_z = A - P_1 + q(z+dz). \tag{b}$$

Subtracting (a) from (b) we obtain $dQ_z = qdz$.
Hence

$$q = \frac{dQ_z}{dz}, \tag{6.1}$$

i.e., *the derivative of the shearing force with respect to the abscissa of a section of a beam is equal to the intensity of distributed load.*

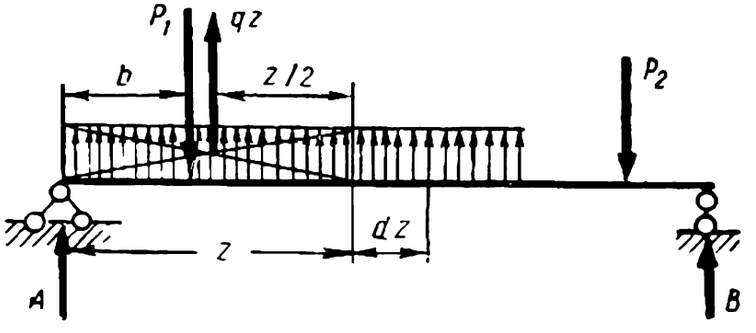


Fig. 6.9

Let us now calculate the bending moment at the section having the abscissa z by summing up the moments of the forces applied to the left of the section. For that the load distributed over the portion of length z is replaced by its resultant equal to qz and applied at the middle of the portion at a distance $\frac{z}{2}$ from the section:

$$M_z = Az - P(z-b) + qz\frac{z}{2}. \tag{c}$$

* The symbol Q_z used here (and M_z below) implies that the shearing force and the bending moment are functions of the abscissa z of a cross section of the beam.

Similarly, calculate the bending moment at an adjacent section at a distance $z+dz$ from the left support:

$$M_z + dM_z = A(z+dz) - P_1(z+dz-b) + \frac{q(z+dz)^2}{2}. \quad \text{(d)}$$

Subtracting (c) from (d) we obtain the magnitude of the increment in the bending moment:

$$dM_z = A\,dz - P\,dz + qz\,dz = dz(A - P_1 + qz).$$

The expression in brackets represents the shearing force Q_z. Consequently

$$dM_z = Q_z\,dz,$$

whence

$$Q_z = \frac{dM_z}{dz}, \tag{6.2}$$

i.e., *the derivative of the bending moment with respect to the abscissa of a section of a beam is equal to the shearing force (Zhuravsky's theorem).*

Differentiating both parts of equality (6.2) we obtain

$$\frac{dQ_z}{dz} = \frac{d^2M_z}{dz^2} = q, \tag{6.3}$$

i.e., *the second derivative of the bending moment with respect to the abscissa of a section of a beam is equal to the intensity of distributed load.*

The relations obtained will be used in what follows, primarily in constructing bending moment and shearing force diagrams.

51. Construction of Bending Moment and Shearing Force Diagrams

For a visual representation of the variation of the bending moment and the shearing force along the length of the beam and for locating the dangerous sections the M_b and Q diagrams are plotted. The procedure for constructing such diagrams will be illustrated by the following examples.

Example 6.4. Draw the M_b and Q diagrams for the beam shown in Fig. 6.10.

Solution. Make a section to the right of the force at a distance z_1 from the right end of the beam (section *I-I*).

The bending moment at this section can be found most easily by summing up the moments of the external forces lying to the right of the section. We obtain $M_b=0$. This result is valid for all sections in the portion BC.

The bending moment at any section II-II in the portion AB can also be calculated as the sum of the moments of all forces lying to the right of the section; then there will

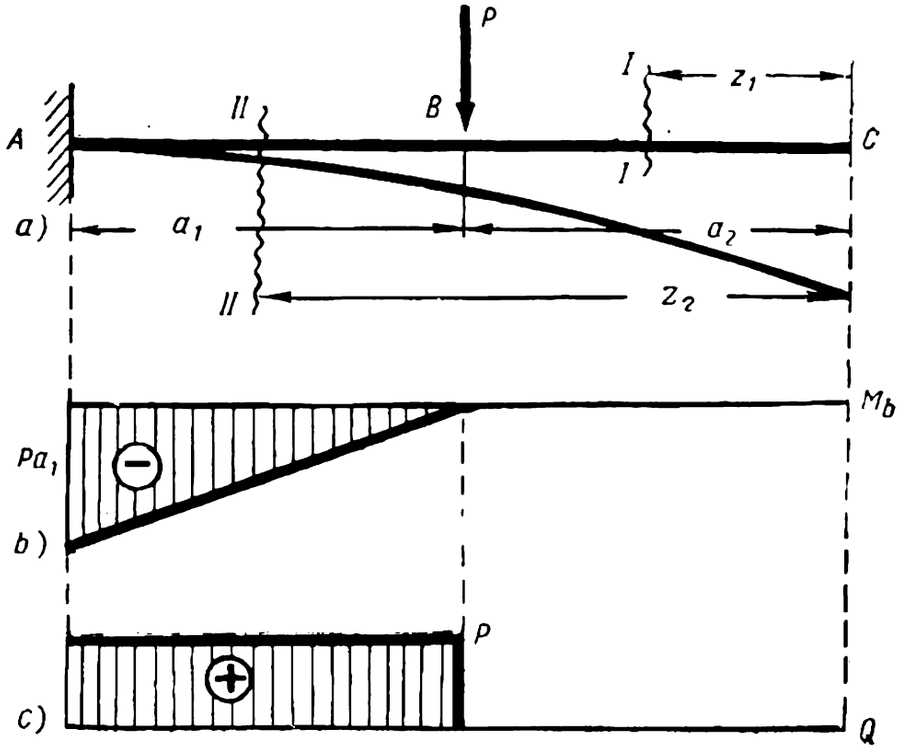


Fig. 6.10

be no need to determine the reactions at the built-in end. We obtain

$$M_{z_2} = -P(z_2 - a_2); \quad (a_2 \leqslant z_2 \leqslant a_1 + a_2).$$

The minus sign is taken because the beam bends concave downward. We have obtained the equation of an inclined straight line. For constructing the diagram we calculate two values of M_{z_2}:

$$M_{z_2=a_2} = 0; \quad M_{z_2=a_1+a_2} + a_2 = -Pa_1.$$

The magnitude of Pa_1 is laid off to a chosen scale below the base line of the diagram. The M_b diagram is presented in Fig. 6.10*b*. The maximum bending moment occurs at the section over the built-in end: $M_{\max} = -Pa_1$.

Now calculate the shearing force at the section I-I. Projecting the forces lying to the right of the section on the

vertical axis we find that $Q_{z_1}=0$. In a similar way we obtain $Q_{z_2}=P$ for section *II-II*. The plus sign is taken because the external force to the right of the section acts downward. The Q diagram is shown in Fig. 6.10*c*.

To determine the sign of the shearing force the following rule may also be proposed along with the foregoing rule: shearing force Q is positive in those portions of the beam where the M_b diagram rises (from left to right) and it is negative in those portions where the M_b diagram falls.

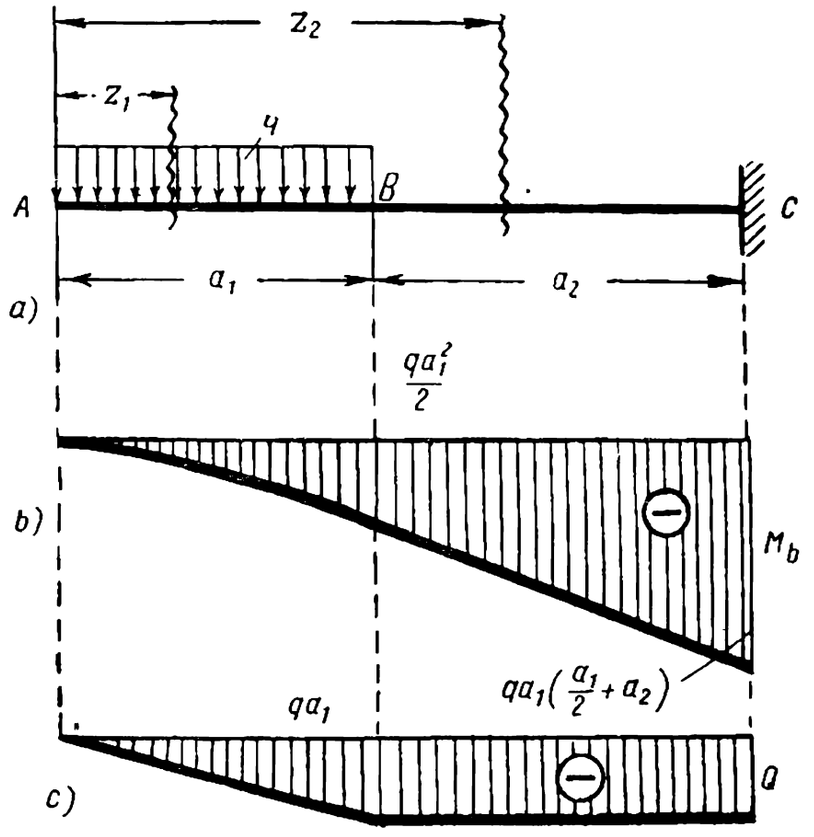


Fig. 6.11

Example 6.5. Plot the M_b and Q diagrams for a cantilever beam (Fig. 6.11*a*).

Solution. There are two portions here (*AB* and *BC*) which differ in type of loading and, consequently, in the laws governing the variation of M_b and Q.

The bending moment in portion *AB* at a section taken at a distance z_1 from point *A* will be found as the moment of the left-hand forces, the distributed load to the left of the section being replaced by its resultant qz_1 applied at the middle of the portion of length z_1. We obtain

$$M_{z_1} = -qz_1\frac{z_1}{2} = -\frac{qz_1^2}{2}.$$

The minus sign is taken because the beam bends concave downward.

This is an equation of a parabola. The parabola is constructed approximately according to three points:

$$M_{z_1=0}=0, \quad M_{z_1=\frac{a_1}{2}}=-\frac{qa_1^2}{8}, \quad M_{z_1=a_1}=-\frac{qa_1^2}{2}.$$

Make a section in the portion BC at a distance z_2 from the free end of the beam. The distributed load over the length a_1 is replaced by its resultant qa_1 applied at the middle of the portion AB. The moment at the section z_2 is

$$M_{z_2}=-qa_1\left(z_2-\frac{a_1}{2}\right).$$

This is an equation of a straight line. Calculate M_{z_2} for two values of z_2: at $z_2=a_1$ and $z_2=a_1+a_2$. We obtain

$$M_B=-\frac{qa_1^2}{2},$$

$$M_C=-qa_1\left(\frac{a_1}{2}+a_2\right).$$

The M_b diagram is given in Fig. 6.11*b*.

The shearing force at the section z_1 as the sum of the projections on the vertical line of the forces to the left of the section is

$$Q_{z_1}=-qz_1.$$

It can also be found by using formula (6.2):

$$Q_{z_1}=\frac{dM_{z_1}}{dz_1}=-qz_1.$$

The shearing force at the section z_2 is $Q_{z_2}=-qa_1$.

The Q diagram is given in Fig. 6.11*c*.

In both cases the shearing force is taken with the minus sign because the M_b diagram is falling. The following relation resulting from formula (6.2) should also be emphasised. In those portions of the beam where the bending moment varies according to a parabola (a curve of the second order) the shearing force varies according to a linear law, i.e., the diagram is an inclined straight line (a line of the first order). Wherever M_b varies according to a linear law, i.e., the M_b diagram is an inclined straight line,

shearing force Q is constant, the diagram is a horizontal straight line (a line of the zero order). In general the function describing a law governing the variation of Q is one order lower than the function which expresses the law governing the variation of M_b. This follows directly from formula (6.2).

Example 6.6. Plot the M_b and Q diagrams for a simply supported beam subjected to a uniformly distributed load (Fig. 6.12).

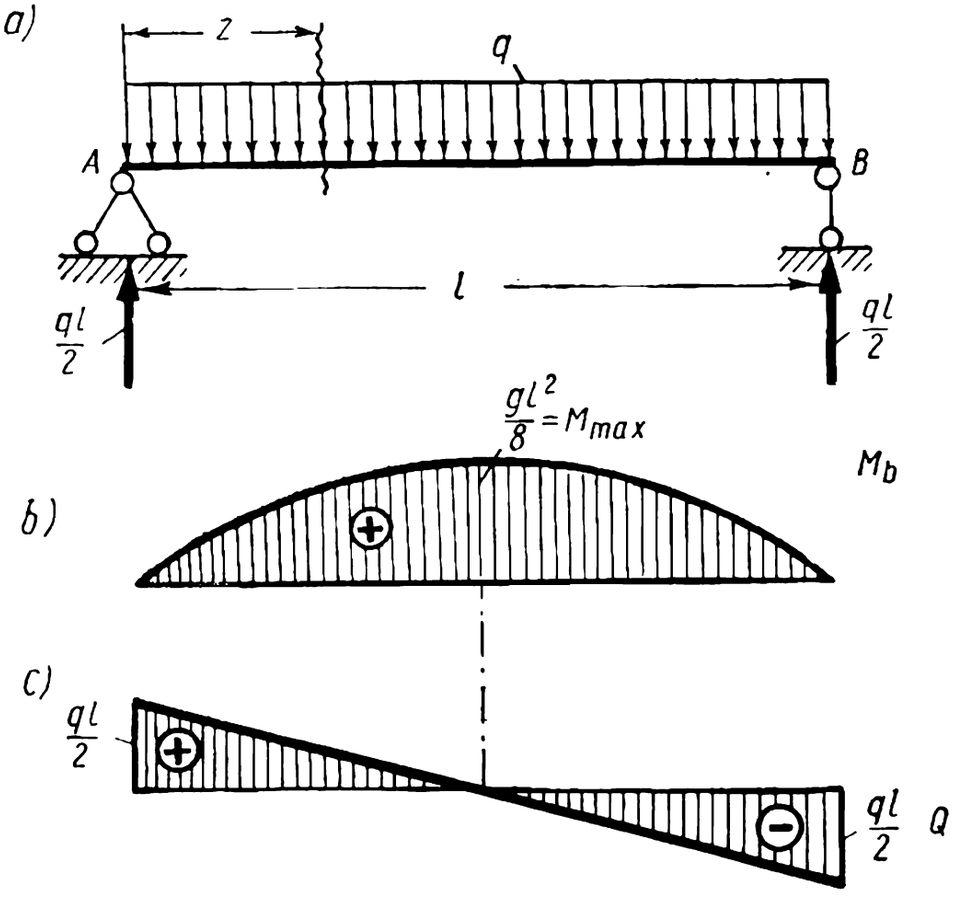


Fig. 6.12

Solution. Because of symmetry of loading of the beam the reactions are equal to each other:

$$A = B = \frac{ql}{2}.$$

The bending moment at the section having an abscissa z is

$$M_z = \frac{ql}{2} z - \frac{qz^2}{2}.$$

The first term represents the bending moment due to the reaction, taken with the plus sign because if we imagine

the beam fixed at the section considered we can see that owing to the action of the supporting force the portion of the beam to the left of the section will tend to bend concave upward. The second term represents the bending moment due to the uniformly distributed load lying to the left of the section made. The resultant of this load is qz and it is applied at the middle of the portion, i.e., at a distance $\frac{z}{2}$ from the section. Consequently, the moment of this load is equal to $\frac{qz^2}{2}$ with the minus sign, as such a load will bend the beam (if the beam is imagined to be fixed at the section) concave downward. The equation obtained for the bending moment is an equation of a parabola. Calculate three ordinates of the M_b diagram:

$$M_{z=0}=0; \quad M_{z=\frac{l}{2}}=\frac{ql^2}{8}; \quad M_{z=l}=0.$$

The M_b diagram is constructed according to these data. The maximum bending moment (at the middle of the beam) is

$$M_{\max}=\frac{ql^2}{8}.$$

It is useful to memorise this result because it is frequently used in calculations. Note that having differentiated the expression for M_z and equated the first derivative to zero we can see that the maximum of M_b actually occurs at mid-span of the beam.

The shearing force at the section z (as the sum of the left forces) is

$$Q_z=\frac{ql}{2}-qz.$$

The same result is obtained by the use of formula (6.2).

Calculate two values of Q_z:

$$Q_{z=0}=\frac{ql}{2}; \quad Q_{z=l}=-\frac{ql}{2}.$$

Construct the Q diagram. Note that $Q=0$ at mid-span of the beam, i.e., where the bending moment is a maximum, in full accord with relation (6.2).

Example 6.7. Plot the M_b and Q diagrams for a cantilever beam (Fig. 6.13).

Solution. The bending moment in portion DC (as the sum of the moments of the left forces) is $M_z=0$.

In portion CB we have: $M_z=-2$ tm. In portion BA we have: $M_z=-2+3=1$ tm. At all sections the shearing force $Q=0$.

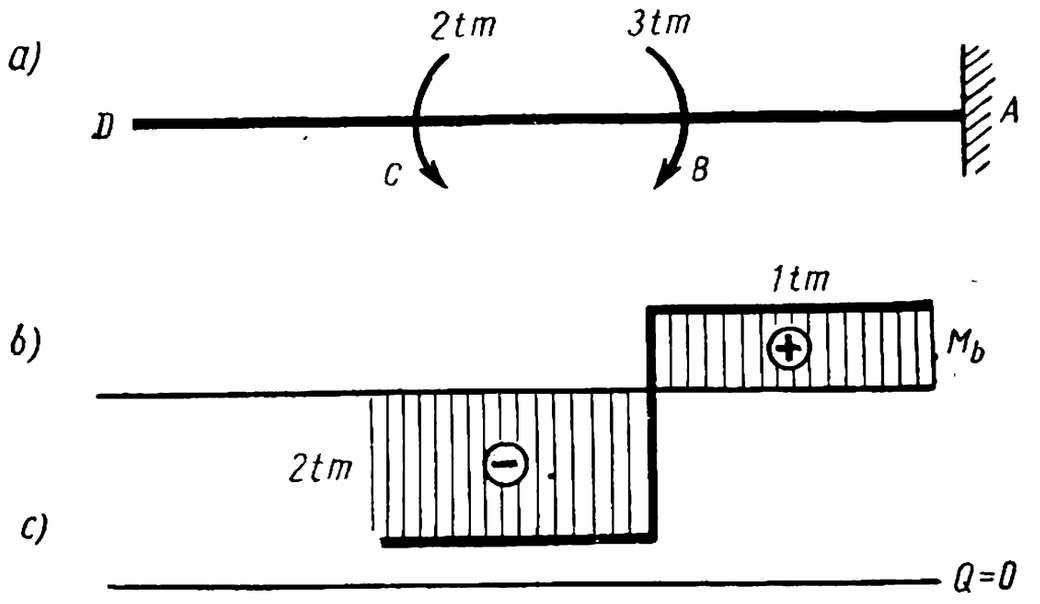


Fig. 6.13

Example 6.8. Plot the M_b and Q diagrams for the beam represented in Fig. 6.14.

By using the equations of equilibrium the reactions are found to be: $A=2$ t downward and $B=14$ t upward.

The bending moment at the section having an abscissa z_1 is determined as the sum of the moments of the left forces:

$$M_{z_1}=-2z_1-\frac{2z_1^2}{2}.$$

This is an equation of a parabola. Calculate three values of M_{z_1}:

$$M_{z_1=0}=0;\quad M_{z_1=1\text{m}}=-3\text{ tm};\quad M_{z_1=2\text{m}}=-8\text{ tm}.$$

Construct the M_b diagram in the portion AE according to these data. Determine the bending moment at the section having an abscissa z_2:

$$M_{z_2}=-2z_2-4(z_2-1)+8.$$

The second term of this expression represents the bending moment due to the resultant of the distributed load acting

over portion AE. This resultant is equal to 4 t and its distance from the section considered is $z_2 - 1$ m.

By putting z_2 equal to $z_2=2$ m and $z_2=4$ m we obtain

$$M_{z_2=2}=0, \quad M_{z_2=4}=-12 \text{ tm}.$$

According to these data the M_b diagram is constructed in the portion EB. Determine the bending moment at the section distant z_3 from the right end of the beam. Since there are fewer external forces to the right of the above section

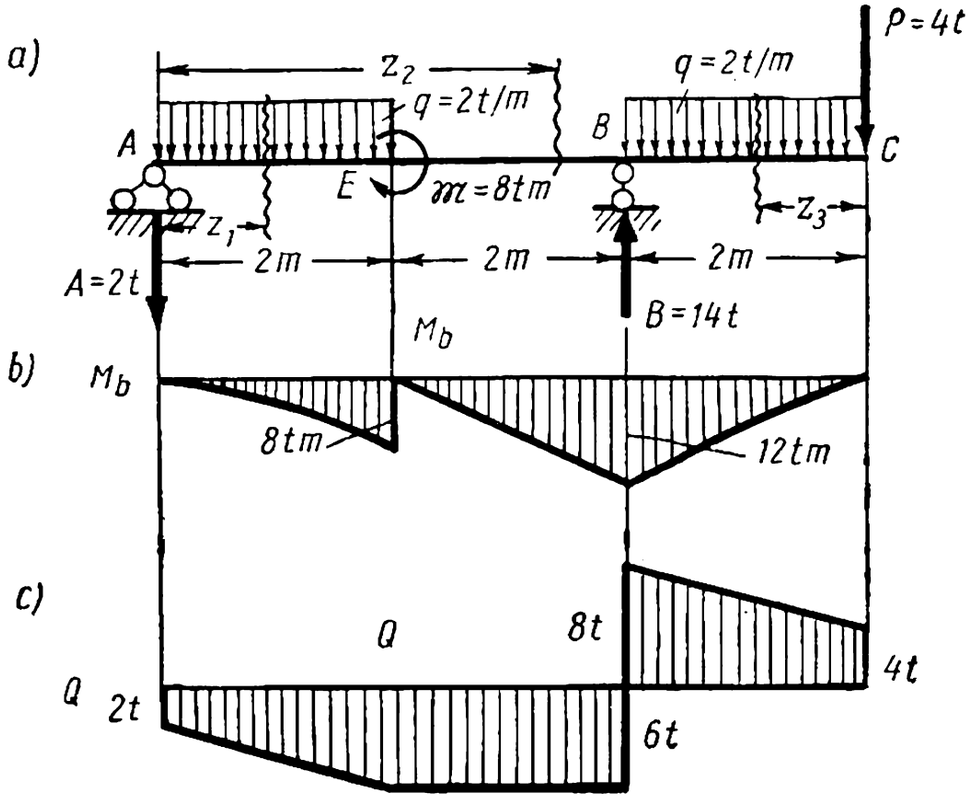


Fig. 6.14

than to the left, it is easier to calculate M_{z_3} as the sum of the moments of the right forces:

$$M_{z_3}=-4z_3-2\frac{z_3^2}{2}.$$

The first term represents the bending moment due to the force $P=4$ t, and the second due to the distributed load acting to the right of the section considered.

Both terms are taken with the minus sign, as the beam, if imagined to be fixed at the section where M_b is being determined, would bend concave downward under the action of the loads to the right of the section.

By putting $z_3=0$ and $z_3=2$ m we obtain the M_b diagram for the portion BC.

Determine the shearing force either by using the relation $Q_z=\frac{dM_z}{dz}$ or by projecting on the vertical line the forces acting on the cut portion. It is recommended to use both methods for checking calculations.

Differentiating M_{z_1} we obtain

$$Q_{z_1}=\frac{d}{dz_1}\left(-2z_1-\frac{2z_1^2}{2}\right)=-2-2z_1.$$

This is an equation of a straight line. The same result is obtained by projecting on the vertical line the forces lying to the left or to the right of the section having the abscissa z_1. The shearing force at any section in the portion EB is

$$Q_{z_2}=\frac{dM_{z_2}}{dz_2}=-6\text{ t}.$$

The Q diagram in the second portion is a horizontal straight line.

In calculating Q_{z_3} as a derivative of the moment M_{z_3} it should be kept in mind that z_3 is measured from right to left and the sign should therefore be changed after differentiation to obtain the correct sign of Q:

$$Q_{z_3}=4+2z_3.$$

The Q diagram is constructed for portion BC according to this equation of a straight line. Discontinuities in the Q diagram are equal in magnitude to the concentrated loads applied at the corresponding sections of the beam: the reactions A and B as well as the force P (at the right end).

Example 6.9. Draw the diagrams of M_b, Q and the longitudinal force N for the broken beam shown in Fig. 6.15*a*.

Solution. The general method of determining M_b, Q and N at any section is the same. However, it is necessary to agree here upon the rule for constructing diagrams for vertical and inclined bars. It is the custom to construct the M_b diagrams for all bars on the concave side of the bar (on the compressed fibre), i.e., to follow the same rule as adopted in constructing diagrams for horizontally located bars. The bending moment at section *I-I* calculated as

the sum of the moments of the external forces lying on one side of the section (beneath it) is

$$M_{z_1} = P z_1.$$

If the lower cut portion of the bar is imagined to be fixed at section *I-I*, it becomes evident that bending will be concave to the left, i.e., the compressed fibre is on the left. Therefore the M_{z_1} diagram is constructed on the left (Fig. 6.15*b*). At $z_1 = 0$ $M_b = 0$; at $z_1 = 2$ m $M_b = 4$ tm.

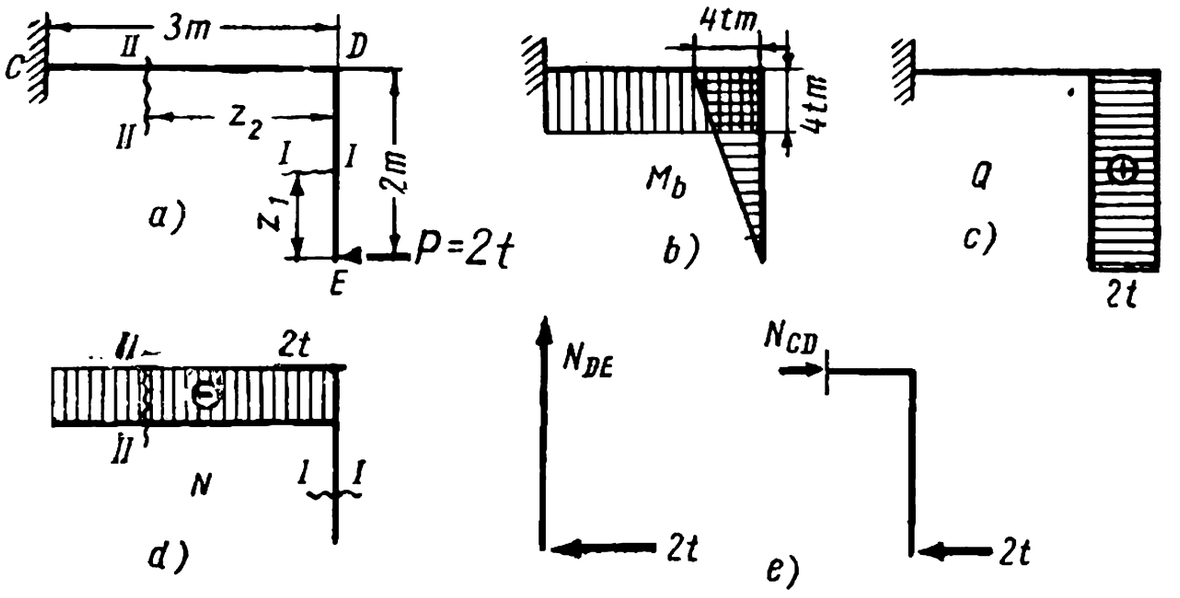


Fig. 6.15

The bending moment at section *II-II* as the sum of the moments of the forces to the right of the section is equal to the product of the force by the distance from the section, i.e., by the length of the segment *DE*:

$$M_{z_2} = 2 \times 2 = 4 \text{ tm}.$$

The compressed fibre is below (see Fig. 6.15*b*).

The shearing force can be determined by using relation (6.2):

$$Q_z = \frac{dM_z}{dz},$$

i.e., Q can be determined as the slope of the tangent to the moment diagram. For the bar *CD* the bending moment $M = \text{const}$, therefore, $Q = 0$. For the bar *DE*

$$Q = \frac{4}{2} = 2 \text{ t}.$$

It is possible, of course, to determine the shearing force as the sum of the projections of the forces acting on one side of the section on the direction perpendicular to the axis of the bar.

The sign of the shearing force is established according to the previous rule. If the M diagram is rising when the bar DE is viewed from left or right, Q is positive. The diagram of positive values of Q is plotted on the right (Fig. 6.15*c*).

The longitudinal force N is determined by applying the method of sections (Fig. 6.15*d*).

For the bar DE (Fig. 6.15*e*) by projecting the forces applied below section *I-I* on the direction of the axis of the bar DE we obtain $N_{DE}=0$.

For the bar CD by projecting the forces applied to the right of section *II-II* on the direction of the axis of the bar CD we obtain $N_{CD}=-2$ t (compression).

The negative values of N are plotted below the axis of the bar. The minus sign signifies compression (see Fig. 6.15*d*).

52. Determination of Normal Stresses

In pure bending only bending moments occur at cross sections of the beam, which act in a plane passing through one of the principal centroidal axes of inertia of the cross section of the beam.

The bending moment represents the resultant moment of internal forces distributed over the section.

The equations of statics are not sufficient to establish the law of distribution and the magnitude of internal forces occurring at the cross section of the beam. It is necessary to use the conditions of deformation of the beam.

If a beam (specimen) with a network of lines marked out on its surface is subjected to pure bending, the following will be discovered (Fig. 6.16):

1. The lines *1-1* and *2-2* on the surface of the beam will rotate through a certain angle $d\varphi$ after deformation and will remain straight. It may be assumed that *cross sections of the beam which are plane before deformation will remain plane after deformation as well (the hypothesis of plane sections)*. Calculations based on this assumption are in full agreement with experiments.

2. Fibre *ab* on the convex side of the beam elongates, which indicates extension of this fibre, whereas fibre *ef* shortens, indicating compression. The length of the fibre *cd*, however, will remain unchanged, which indicates that this fibre undergoes neither extension nor compression.

The layer of the beam (at the level of the fibre *cd*) which does not undergo strain in bending is called *the neutral layer*. The line of intersection of the neutral layer with the plane of the cross section of the beam (Fig. 6.17) is called *the neutral axis (line)*. The intersection of the plane of loading with the plane of the cross section is called *the line of loading*.

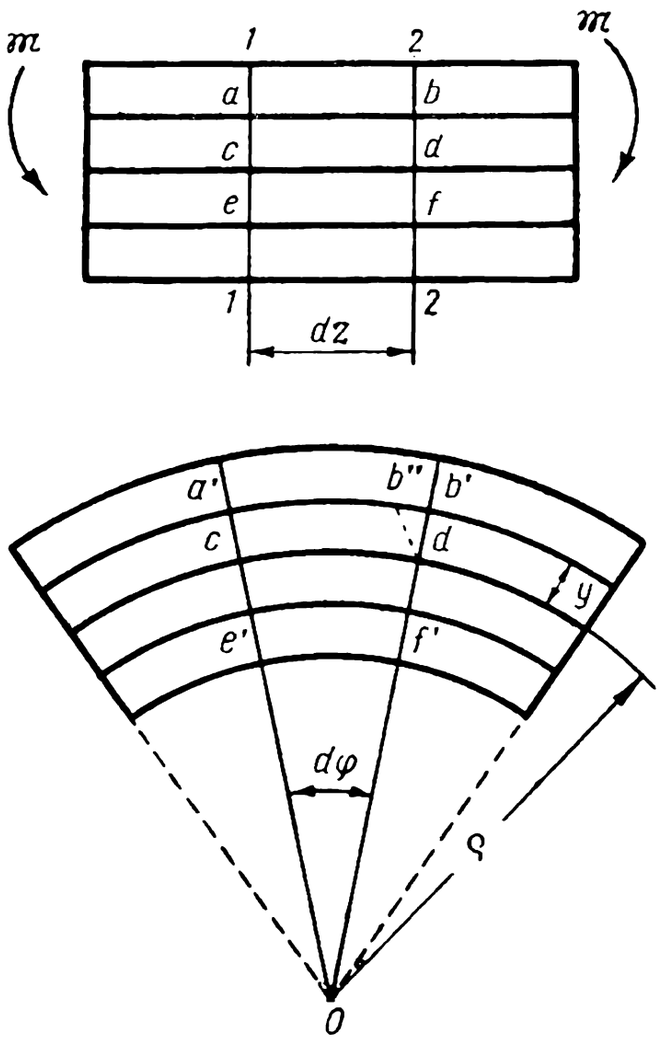


Fig. 6.16

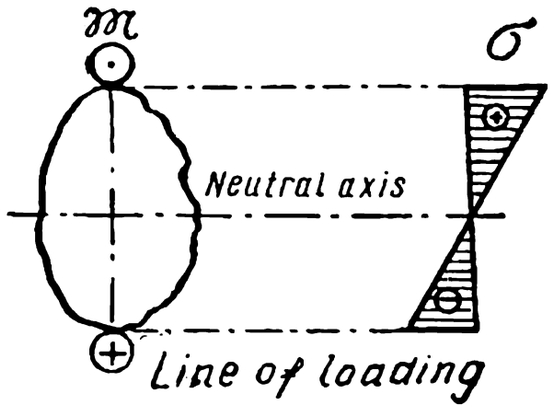


Fig. 6.17

From the experimental results considered it follows that the fibres of the beam deform in different ways: those which are farther from the neutral layer undergo greater deformations. We shall show that deformations vary according to a linear law across the depth of the beam.

Indeed, the segment $b'b''$ represents the total elongation of the fibre *ab*, whose length before deformation is equal to the length of the fibre *cd* belonging to the neutral layer. The unit elongation of this fibre is

$$\varepsilon = \frac{b'b''}{ab} = \frac{b'b''}{cd} = \frac{yd\varphi}{\rho d\varphi} = \frac{y}{\rho}, \qquad (6.4)$$

where ρ is the radius of curvature of the neutral layer of the beam; the magnitude of ρ is unknown as yet; y is the distance from the neutral axis to the fibre being considered.

Before proceeding to the determination of stresses let us introduce one more hypothesis, namely, assume that the fibres of the beam do not exert pressure upon each other, i.e., stresses in the direction perpendicular to the axis of the beam are equal to zero. Consequently, each fibre undergoes uniaxial tension or compression. The formula obtained on the basis of this hypothesis yields results which are in good agreement with experimental data. Then, according to Hooke's law for the uniaxial state of stress we obtain

$$\sigma = E\varepsilon = E\frac{y}{\rho}, \tag{6.5}$$

i.e., *normal stresses vary across the depth of the cross section of the beam directly as the distance from the neutral axis.* The maximum stresses occur at the upper and lower edges of the section.

The σ diagram is shown in Fig. 6.17. Tensile stresses are considered to be positive.

It should be pointed out that the normal stress vectors are, of course, perpendicular to the plane of the cross section of the beam, and the segments representing them in the diagram are shown as if coinciding with the plane of the section.

After establishing the law of stress distribution, it is possible to determine the magnitude of stresses from the equations of equilibrium. Consider equilibrium of a portion of the beam subjected to the external moment $\mathfrak{M}$ and the internal forces developed at the above cross section (Fig. 6.18). Since this portion of the beam is in equilibrium six equations of equilibrium must be satisfied: the sums of the projections of the acting forces on three coordinate axes must be zero and three sums of the moments about the x, y and z axes must be zero:

$$1.\ \Sigma Y = 0, \qquad 2.\ \Sigma X = 0$$

are identically zero as the internal forces σdF are perpendicular to these axes;

$$3.\ \Sigma Z = 0 \quad \text{or} \quad \int_F \sigma\, dF = 0.$$

By using formula (6.5) we obtain

$$\frac{E}{\rho}\int_F y\,dF = 0.$$

But $\frac{E}{\rho} \neq 0$ since $\rho \neq \infty$ for the deflected beam being considered. Therefore, $\int_F y\,dF = 0$.

This integral represents the statical moment of the area of the cross section of the beam with respect to the neutral axis. It is equal to zero, consequently, *the neutral axis in bending passes through the centroid of the section.*

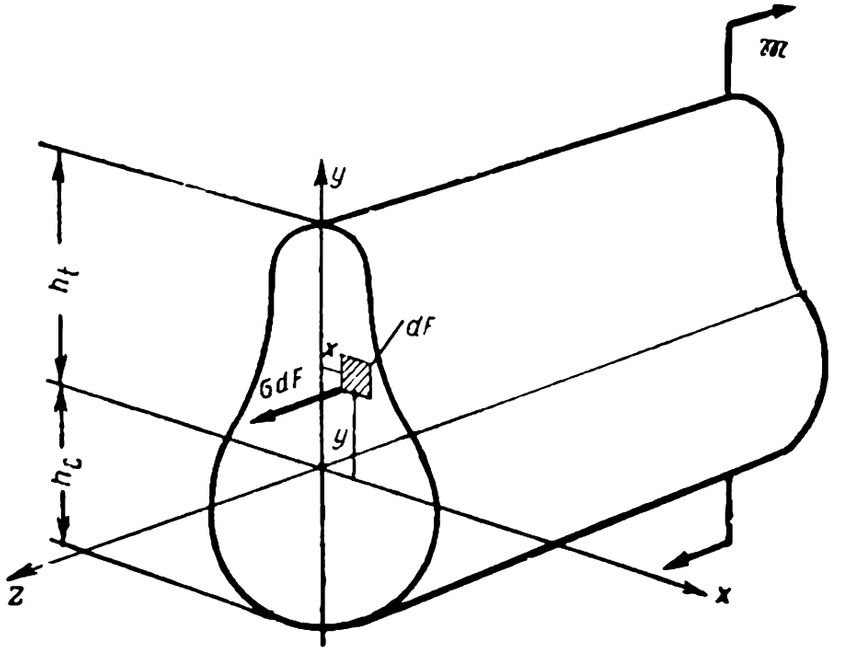


Fig. 6.18

4. The equation $\Sigma M_z = 0$ is identically zero as the internal forces σdF are parallel to the z-axis.

5. The equation $\Sigma M_y = 0$ yields $\int_F \sigma\,dF x = 0$. By using formula (6.5) we obtain

$$\frac{E}{\rho}\int_F xy\,dF = 0.$$

But $\frac{E}{\rho} \neq 0$, therefore,

$$\int_F xy\,dF = 0.$$

The integral $J_{xy}=\int\limits_F xy\,dF$ represents the product of inertia of the cross section with respect to the x and y axes.

Since it is equal to zero, the x and y axes must be principal axes of inertia of the section and the moment $\mathfrak{M}$ must lie in a plane passing through one of the principal axes, which is the case in plane bending. From this condition it also follows that *the line of loading and the neutral axis (zero line) are mutually perpendicular.*

6. We equate to zero the sum of the moments of the forces about the x-axis

$$\Sigma M_x=0, \qquad -\mathfrak{M}+\int\limits_F y\sigma\,dF=0.$$

By using formula (6.5) we obtain

$$\mathfrak{M}=\frac{E}{\rho}\int\limits_F y^2\,dF.$$

The integral $J_x=\int\limits_F y^2\,dF$ represents the moment of inertia of the cross section about the neutral x-axis.

There may be more than one external couple as well as any other load acting on the cut portion of the beam. In this case the equation of equilibrium $\Sigma M_x=0$ will include the algebraic sum of the moments of all these forces, equal in magnitude to the bending moment at the cross section.

Bearing in mind the above, the latter relation becomes

$$M_b=\frac{E}{\rho}J_x, \qquad \text{whence} \tag{6.6}$$

$$\frac{1}{\rho}=\frac{M_b}{EJ_x}. \tag{6.7}$$

The quantity $K=\frac{1}{\rho}$ represents the curvature of the neutral layer of the beam.

As shown previously, the neutral line of the cross section passes through its centroid. Therefore, the axis (longitudinal axis) of the beam which is the locus of the centroids of its cross sections lies in the neutral layer. Hence, expression (6.7) determines the curvature of the axis of the beam.

Thus, *the curvature of the axis of the beam in bending is directly proportional to the bending moment and in-*

versely proportional to the quantity EJ_x which is called the flexural rigidity of the section. Substituting the above value of $\frac{1}{\rho}$ in (6.5) we obtain an important formula

$$\sigma = \frac{M_b}{J_x} y \tag{6.8}$$

giving the magnitude of the normal stress at any point of the cross section of the beam provided the bending moment M_b and the moment of inertia of the section are known.

As more detailed investigations show, formula (6.8) is also suitable for determining normal stresses in the general case of plane bending (in transverse plane bending) when both the bending moment and the shearing force act on cross sections of the beam.

53. Condition of Strength Based on Normal Stresses

To ensure strength in a beam it is necessary that the greatest tensile and the greatest compressive flexure stresses at the dangerous section, i.e., at the section where M_b is a maximum,* should not exceed the corresponding allowable stresses.

Let (see Fig. 6.18) h_t denote the distance from the neutral axis to the most remote fibre in tension and h_c that to the most remote fibre in compression. Then the maximum tensile flexure stress will be

$$\max \sigma_t = \frac{M_b}{J_x} h_t, \tag{6.9}$$

the maximum compressive stress (in absolute value)

$$\max \sigma_c = \frac{M_b}{J_x} h_c. \tag{6.10}$$

For brittle materials (such as cast iron) the allowable stresses in tension and compression are different: $[\sigma_c]$ is 3-5 times as large as $[\sigma_t]$, therefore cross sections that are not symmetrical with respect to the neutral axis are generally used for beams of such materials. The section is proportioned so as to have $h_t < h_c$, i.e., to assure the in-

* Only beams of constant cross section along the entire length are under consideration.

equality $\max \sigma_t < \max \sigma_c$. In the above cases two conditions of strength are required:

$$\max \sigma_t = \frac{M_b}{J_x} h_t \leqslant [\sigma_t], \tag{6.11a}$$

$$\max \sigma_c = \frac{M_b}{J_x} h_c \leqslant [\sigma_c]. \tag{6.11b}$$

The maximum numerical value of M_b should be substituted in formulas (6.11*a*), (6.11*b*).

If the cross section of a beam is symmetrical with respect to the neutral axis (it is advisable to use such sections for beams of ductile materials), i.e., $h_t = h_c = \frac{h}{2}$, we obtain one formula in place of two formulas (6.9) and (6.10):

$$\sigma = \frac{M_b h}{J_x \times 2}. \tag{6.12}$$

Denoting $W_x = \frac{2J_x}{h}$ we obtain the following condition of strength if the allowable stresses in tension and compression $[\sigma]$ are equal:

$$\max \sigma = \frac{M_b}{W_x} \leqslant [\sigma]. \tag{6.13}$$

The quantity W_x is called the axial section modulus or the section modulus in bending. The section modulus is a geometrical characteristic of the cross section of a beam which determines its strength in bending.

The values of W_x for the simplest sections are as follows:

(a) for a rectangle $W_x = \frac{2J_x}{h} = \frac{bh^3}{12\frac{h}{2}} = \frac{bh^2}{6}$,

(b) for a circle $W_x = \frac{2J_x}{d} = \frac{\pi d^4}{64d/2} = \frac{\pi d^3}{32} \approx 0.1\, d^3$,

(c) for a ring

$$W_x = \frac{2J_x}{D} = \frac{\pi D^4 (1 - c^4)}{64D/2} = \frac{\pi D^3}{32}(1 - c^4) \approx 0.1\, D^3 (1 - c^4),$$

(d) for rolled sections (I-sections, channels, etc.) values of W_x are set out in standard tables. To select the section of a beam, we obtain the following relation from equation (6.13):

$$W_x \geqslant \frac{M_b}{[\sigma]}. \tag{6.14}$$

The allowable bending moment is given by

$$[M_b] = W_x [\sigma]. \tag{6.15}$$

Having obtained the allowable magnitude of the bending moment according to this formula and knowing the relation between M_b and the load (from the M_b diagram constructed) the magnitude of the allowable load can be determined.

Example 6.10. Select the section of an I-beam having a span $l=6$ m and carrying a uniformly distributed load (see Fig. 6.12):

$$q = 4 \frac{\text{t}}{\text{m}}, \quad [\sigma] = 1{,}600 \text{ kg/cm}^2.$$

Solution. In this case the maximum bending moment will occur at the middle section of the beam:

$$\max M_b = \frac{ql^2}{8} = \frac{4 \times 36}{8} = 18 \text{ tm} = 18 \times 10^5 \text{ kg cm}.$$

The required section modulus is

$$W_x = \frac{\max M_b}{[\sigma]} = \frac{18 \times 10^5}{1{,}600} = 1{,}125 \text{ cm}^3.$$

From tables on profile sections we choose I-section No. 45 which has $W_x = 1{,}220$ cm^3 (according to GOST 8239-56).

In this and subsequent examples use is made of new tables on profile sections (according to GOST 8239-56).

If the international system of units (Si) is employed the solution is as follows.

The distributed load acting on the beam is $q=4$ t/m$=$ $=4 \times 10^4$ N/m. The allowable stress is $[\sigma]=$ $=16 \times 10^7$ N/m$^2 = 160 \frac{\text{MN}}{\text{m}^2}$, where MN designates *meganewton.*

The maximum bending moment is

$$\max M_b = 18 \times 10^5 \text{ kg cm} = 18 \times 10^6 \text{ N cm} = 18 \times 10^4 \text{ N m}.$$

The required section modulus is

$$W_x = \frac{18 \times 10^4 \text{ N m}}{16 \times 10^7 \text{ N/m}^2} = 0.001125 \text{ m}^3 = 1{,}125 \text{ cm}^3.$$

Example 6.11. Determine the allowable load for a beam of rectangular cross section (Fig. 6.19) if $[\sigma]=100$ kg/cm^2.

Solution. Determine the magnitude of the allowable bending moment

$$[\max M_b] = W_x [\sigma] = \frac{bh^2}{6} [\sigma] =$$

$$= \frac{12 \times 24^2}{6} 100 = 115.2 \times 10^3 \text{ kgcm} = 1.152 \text{ tm}.$$

In order to determine the allowable load it is necessary to know the relation between the maximum bending mo-

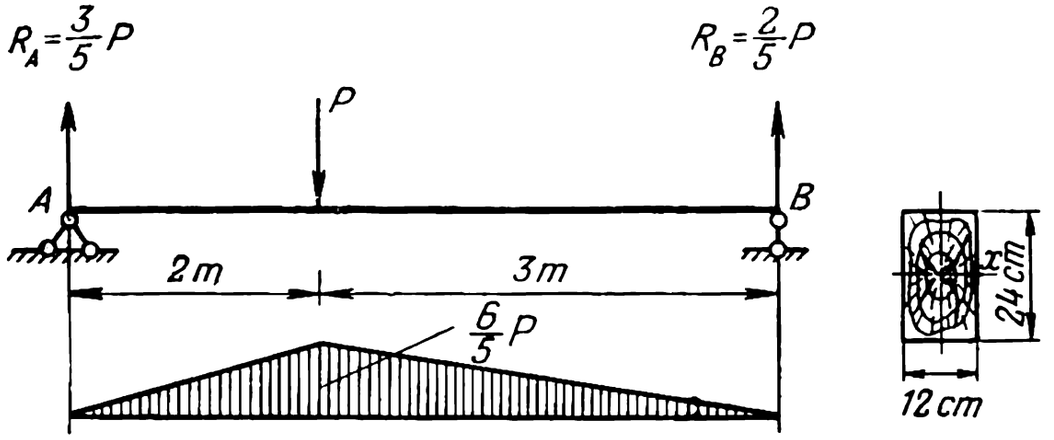


Fig. 6.19

ment and the load, and for this it is necessary to plot the bending moment diagram.

Determining the reactions, we find that $R_A = \frac{3}{5}P$, $R_B = \frac{2}{5}P$. The maximum bending moment occurs at the section under the load and is given by

$$\max M_b = R_A 2 \text{ m} = \frac{6}{5}P.$$

The lengths are taken in metres. We can now determine the allowable load

$$[P] = \frac{5}{6} [\max M_b] = \frac{5}{6} 1.15 \cong 0.96 \text{ t}.$$

54. Determination of Shearing Stresses

In the general case of bending (in transverse bending) bending moments and shearing forces occur at cross sections of a beam. The presence of the bending moment is

due to normal stresses developed on transverse sections of a beam which are determined by formula (6.8).

The presence of the shearing force is due to shearing stresses developed on transverse sections of a beam and according to the law of conjugate shearing stresses on its longitudinal sections as well (Fig. 6.20). To determine shearing stresses consider first a narrow beam of rectangular cross section (Fig. 6.21). Let us cut out from the beam an element of length *dz* and of the same width as

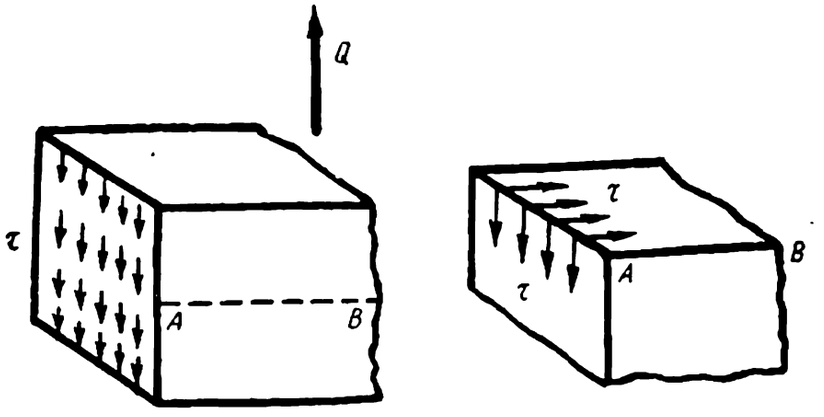


Fig. 6.20

the beam *b*. This element is acted upon by the following forces.

Over the face *344'3'* normal stresses act which, according to formula (6.8), are

$$\sigma_1 = \frac{M_1}{J_x} y, \qquad \text{(a)}$$

where M_1 is the bending moment in the section *344'3'*. Furthermore, shearing stresses τ as yet unknown act on the above section which in view of the narrowness of the section of the beam may be assumed to be uniformly distributed across the width of the section. *

Over the face *122'1'* normal stresses

$$\sigma_2 = \frac{M_2}{J_x} y \qquad \text{(b)}$$

and shearing stresses τ act.

Over the face *322'3'* only shearing stresses act which are equal according to the law of conjugate shearing stresses to shearing stresses acting on the vertical faces. Derive

* This assumption is known as Zhuravsky's hypothesis.

the equation of equilibrium for the cut element of the beam. Project the forces acting upon the element on the horizontal axis. Obviously, the shearing forces acting over the vertical faces will not enter in the above equation.

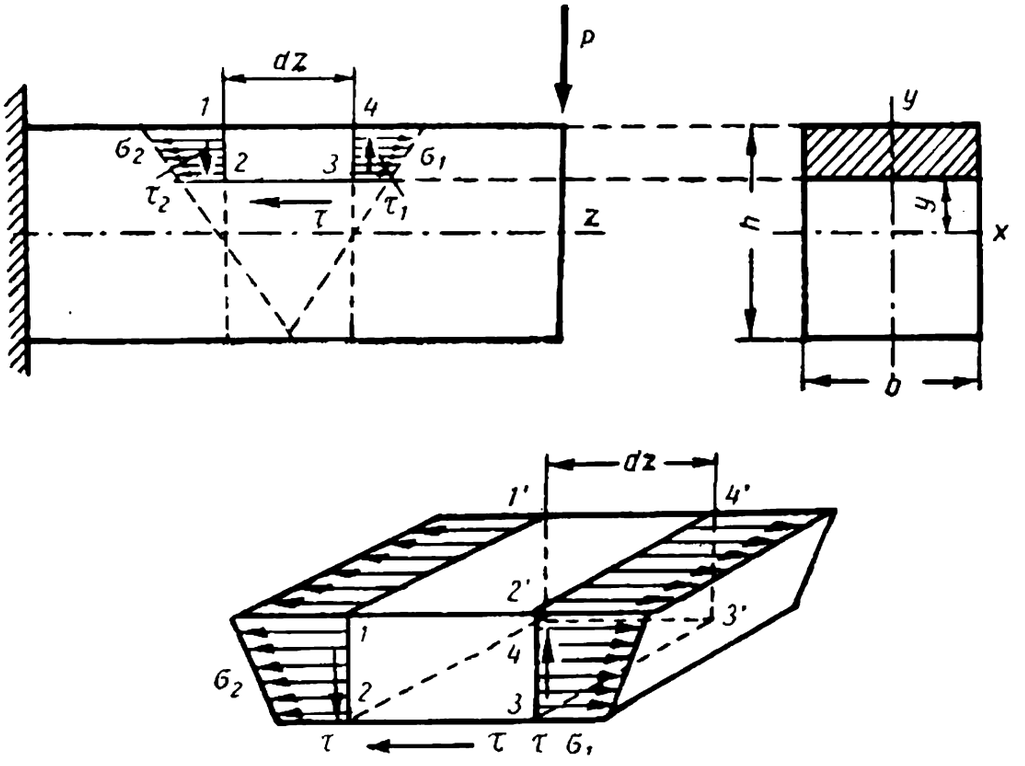


Fig. 6.21

The shearing force over the face *233'2'* will have a projection equal to its true magnitude $\tau b\, dz$. The normal forces acting over the face *344'3'* have the resultant

$$N_1 = \int_{F_{cut}} \sigma_1 \, dF.$$

The normal forces acting over the face *122'1'* have the resultant

$$N_2 = \int_{F_{cut}} \sigma_2 \, dF.$$

The integrals should be extended over the area of the cut portion, i.e., over the area of the faces *122'1'* and *344'3'*. By using the equation of equilibrium $\Sigma Z = 0$ we obtain

$$-N_2 + N_1 - \tau b\, dz = 0$$

or

$$-\int_{F_{cut}} \sigma_2\, dF + \int_{F_{cut}} \sigma_1\, dF - \tau b\, dz = 0.$$

By using expressions (a) and (b) we obtain

$$-\frac{M_2}{J_x}\int_{F_{cut}} y\, dF + \frac{M_1}{J_x}\int_{F_{cut}} y\, dF - \tau b\, dz = 0.$$

The expression $\int_{F_{cut}} y\, dF = S_x^{cut}$ represents the statical moment of the area of the cut portion with respect to the neutral axis. Therefore

$$\frac{S_x^{cut}}{J_x}(M_1 - M_2) = \tau b\, dz.$$

But $M_1 - M_2 = dM_z$ is the increment of the bending moment over the length dz, so that the preceding formula can be rewritten as

$$\frac{S_x^{cut}\, dM_x}{J_x} = \tau b\, dz,$$

whence

$$\tau = \frac{S_x^{cut}\, dM_z}{J_x b\, dz}.$$

By using relation (6.2)

$$\frac{dM_z}{dz} = Q_z$$

we obtain finally

$$\tau = \frac{QS_x^{cut}}{J_x b}. \qquad (6.16)$$

This relation was first established by D. I. Zhuravsky and is called after him.

Investigate the law of shearing stress distribution over the section of a beam of rectangular cross section (Fig. 6.22). This law is determined by the law of variation of S_x^{cut} as the remaining quantities are constant for a given section; we have

$$J_x = \frac{bh^3}{12}.$$

The statical moment of the shaded area with respect to the x-axis is

$$S_x^{cut} = b\left(\frac{h}{2} - y\right)\frac{1}{2}\left(\frac{h}{2} + y\right) = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right)$$

This is an equation of a parabola. The shearing stresses are:

$$\tau = \frac{Qb\left(\frac{h^2}{4} - y^2\right) \times 12}{bh^3 \times 2 \times b} = \frac{6Q}{bh^3}\left(\frac{h^2}{4} - y^2\right).$$

Draw the τ diagram according to three points:

$$\tau_{y=\frac{h}{2}} = 0; \quad \tau_{y=0} = \frac{3}{2}\frac{Q}{F}; \quad \tau_{y=-\frac{h}{2}} = 0.$$

The τ diagram is shown in Fig. 6.22. The maximum shearing stress for a beam of rectangular cross section occurs at the level of the neutral axis and is given by

$$\tau_{max} = \frac{3}{2}\frac{Q}{F}, \tag{6.17}$$

i.e., it is half as much again as the stress which would be obtained on the assumption of uniform distribution of shearing stresses over the section.

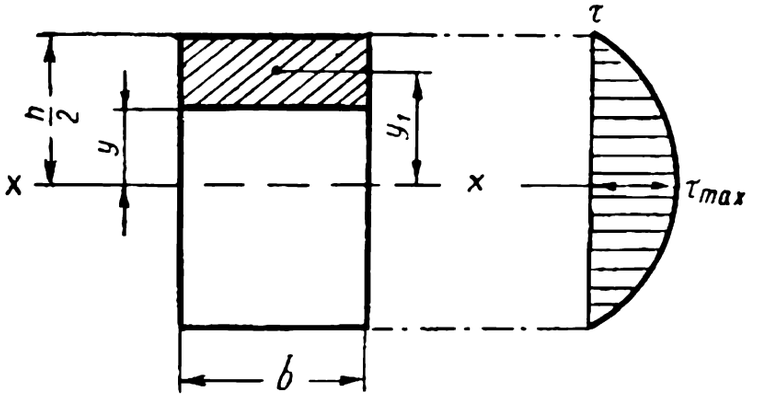


Fig. 6.22

With a certain degree of approximation Zhuravsky's formula may be employed for calculating shearing stresses in beams with other cross-sectional shapes. For a circular cross section one may, using a similar method, obtain the τ diagram shown in Fig. 6.23, with the maximum value at the neutral axis:

$$\tau_{max} = \frac{4}{3}\frac{Q}{F}. \tag{6.18}$$

For a ring cross section

$$\tau_{max} = \frac{2Q}{F}. \tag{6.19}$$

The approximate nature of the foregoing method of determining shearing stresses for such sections as a circle, triangle, etc., becomes evident when we consider the conditions on the contour.

Consider, for instance, a point A near the contour of a circular section (Fig. 6.23).

Applying Zhuravsky's method outlined above, we find the shearing stress τ acting vertically upward. Resolve it into two components, in the direction of the normal to the contour τ_n and in the direction of the tangent τ_t. According to the loading conditions, however, the surface of the bar is free of stresses. Consequently the shearing stress at the point A, as well as at any other contour point, cannot be directed along the vertical; it can be directed only along the tangent to the contour. By using the foregoing method we determine, therefore, only the vertical component of the shearing stress rather than its total magnitude.

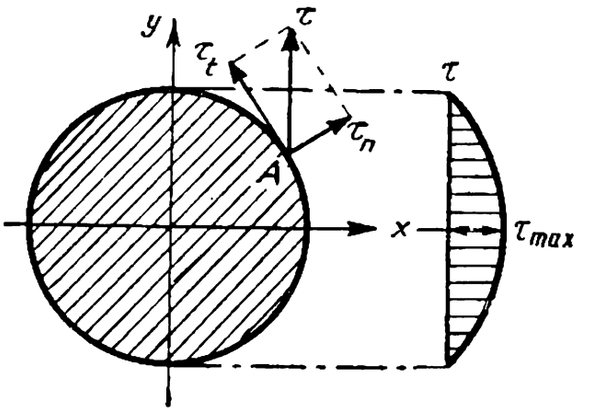


Fig. 6.23

The determination of the horizontal component involves the use of more complicated procedures than those considered above. The analysis of exact solutions of the theory of elasticity shows that in the majority of cases the x-components of τ play a much smaller role than the y-components.

For I-beams the τ diagram has a stepwise form due to an abrupt change in the beam width * (Fig. 6.24a).

The maximum shearing stress in an I-section occurs at the points of the neutral axis and is determined according

* It should be noted that the part of the diagram pertaining to the flanges of the beam is only a conventional representation as the hypothesis of uniform distribution of shearing stresses across the width of the secion does not apply here. Bearing this in mind, the τ diagram in cross sections of I-beams is constructed, as a rule, within the web only.

to Zhuravsky's formula with the statical moment of the shaded area (half-section). Tables on profile sections give values of the statical moment of the half-sections for I-beams and channels. Fig. 6.24 *b* and *c* show the shape of the τ diagrams for some other sections.

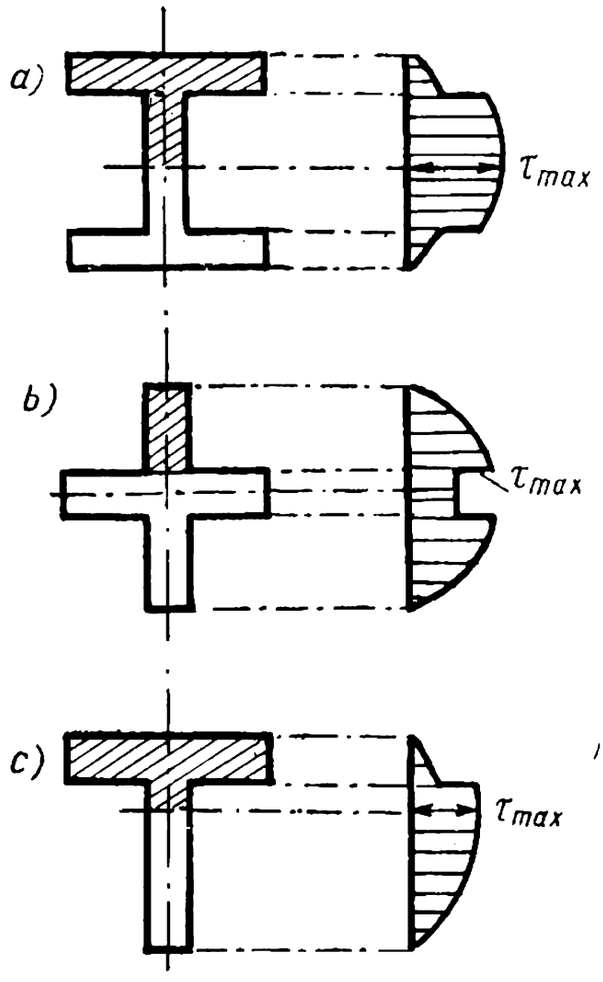


Fig. 6.24

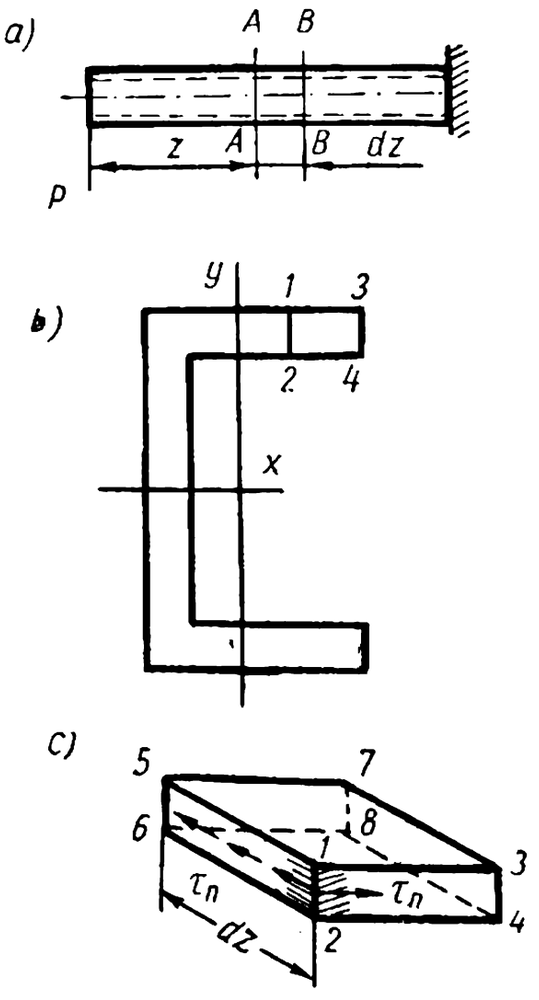


Fig. 6.25

The condition of strength based on shearing stresses is written as

$$\tau_{max} \leqslant [\tau],$$

where $[\tau]$ is the allowable shearing stress. For steel beams $[\tau] \approx 0.6\,[\sigma]$.

Some materials such as wood (parallel to the grain) have very low resistance to shear, and for beams made of such materials a check on strength based on shearing stresses is indispensable.

The foregoing theory for determining shearing stresses is valid only for solid sections.

In thin-walled bars, as has already been pointed out, even when the plane of loading coincides with one of the principal centroidal axes of inertia of the cross section twisting is likely to occur. In order to understand this better, let us consider a cantilever box beam (Fig. 6.25*a*). The cross section of this beam is represented on an enlarged scale in Fig. 6.25*b*.

Let load P lie in a plane passing through the principal centroidal axis of inertia of the section y which is not an axis of symmetry of the section (Fig. 6.25*b*). Cut by a section *12* parallel to the yz plane a portion of the upper flange of length dz and consider the equilibrium of the cut portion (Fig. 6.25*c*). Let the face *1234* of the element belong to section B. The faces *8657* and *1234* of this ele-

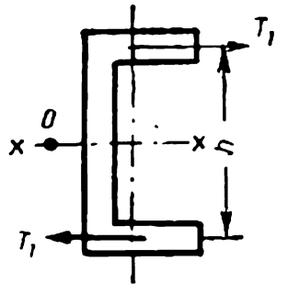


Fig. 6.26

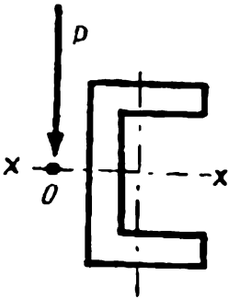


Fig. 6.27

ment are acted upon by normal stresses σ_1 and σ_2. The normal force acting on the area *1234* lying in section B is greater than the normal force acting on the area *8657* as the bending moment at section B is greater than at section A.

Thus, the isolated element can be held in equilibrium only if shearing stresses τ_n act over the face *1265*. But we have established the law of conjugate shearing stresses, and therefore if shearing stresses occur on the face *1265* there must occur shearing stresses equal in magnitude and opposite in sign at the points of the rib *12* on the area *1243*. By similar reasoning we can see that in the lower flange of our box beam there will also occur horizontal shearing stresses the direction of which will be opposite to that of the shearing stresses acting in the sections of the upper flange.

The resultants of these forces T_1 form an internal couple T_1h (Fig. 6.26), an "internal torque".

Thus, the bending of the bar is accompanied by twisting. The distribution of normal stresses at a section will in this case follow a more complicated law than in plane bending.

To obtain plane bending with the neutral axis x the vertical plane in which the force P is acting must pass through a certain point O (Fig. 6.27) called *the centre of bending* (variously called the centre of twist, the centre of rigidity, the shear centre). The theory of analysis of thin-walled bars in bending and in torsion was developed by Prof. V. Z. Vlasov.

55. Stresses on Inclined Sections of a Beam. Principal Stresses

We have established that normal and shearing stresses act on transverse sections of a beam whereas only shearing stresses occur on longitudinal sections. * On inclined sections of a beam, for instance, over the area *bc*

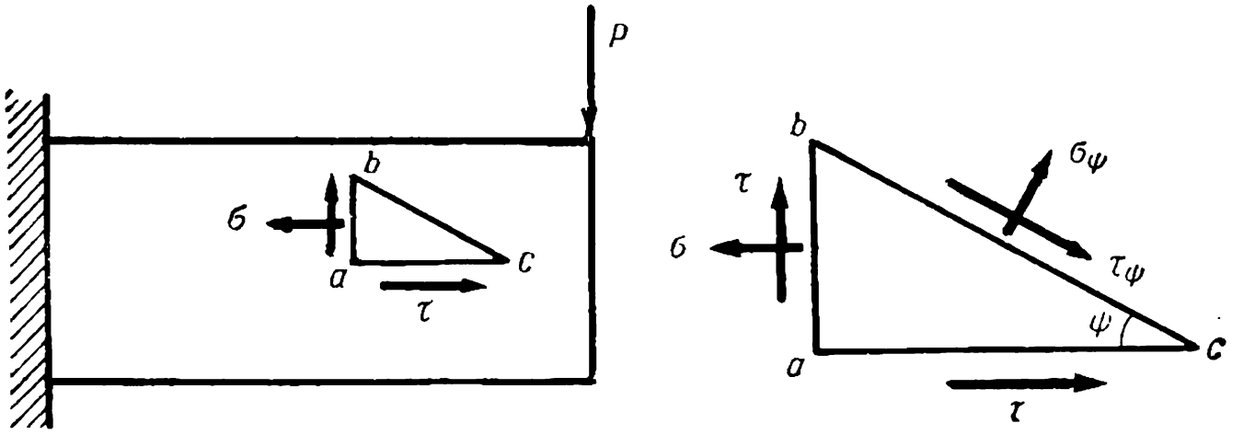


Fig. 6.28

(Fig. 6.28) both normal and shearing stresses are produced which can be calculated by the use of formulas in Section 19.

For determining the principal stresses let us make use of formula (2.37):

$$\sigma_{\substack{\max \\ \min}} = \frac{\sigma}{2} \pm \frac{1}{2} \sqrt{\sigma^2 + 4\tau^2}.$$

* The absence of normal stresses on longitudinal sections of a beam follows from the hypothesis adopted concerning the absence of mutual pressure between fibres (see Section 52).

The angle of inclination of the principal areas is determined by formula (2.35):

$$\tan 2\psi_0 = \frac{2\tau}{\sigma}.$$

The maximum shearing stresses acting on the areas making an angle $\pm 45°$ with the principal areas according to formula (2.32) are

$$\tau_{\substack{\max \\ \min}} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \pm \frac{1}{2} \sqrt{\sigma^2 + 4\tau^2}.$$

From these formulas it is seen that the principal stresses $\sigma_{\substack{\max \\ \min}}$ and $\tau_{\substack{\max \\ \min}}$ will have the greatest value at those points in the cross section of a beam where considerable normal and shearing stresses are present simultaneously. For an I-beam, for instance, such a point is point A, i.e., the top (or the bottom) point of the web (Fig. 6.29).

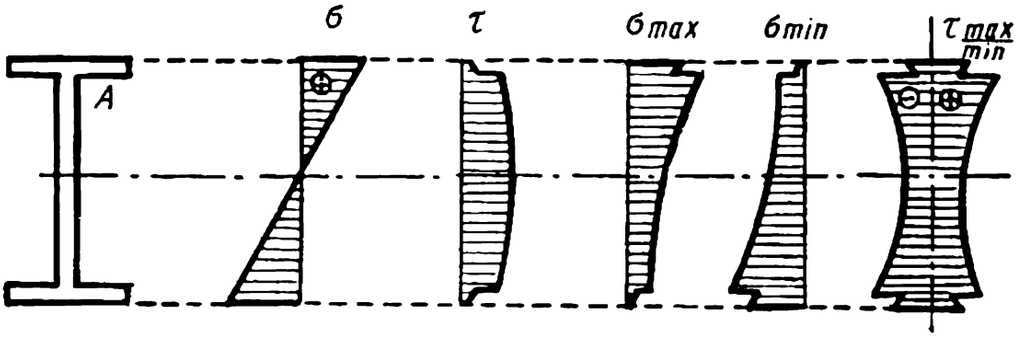


Fig. 6.29

In the same figure approximate diagrams are shown for an I-section for σ_{max}, σ_{min}, τ_{max} and τ_{min}, drawn on the basis of the above formulas.

It is also evident that the principal stresses should be checked in those sections along the length of a beam where both the bending moment (determining σ) and the shearing force (determining τ) have great values simultaneously.

56. Stress Concentration in Bending

At places where there is a sharp change in the contour of the longitudinal section of a beam the picture of stress distribution changes abruptly due to stress concentration (Fig. 6.30*a*).

To reduce stress concentration sharp changes in the contour of the longitudinal section of a beam should be avoided by using smooth transitions, the so-called fillets (Fig. 6.30*b*).

The influence of stress concentration on the static strength for low-ductile and brittle materials is evaluated either by the theoretical stress concentration factor α_σ calculated by the methods of the theory of elasticity, or by the effective stress concentration factor k_s determined experimentally. For that the

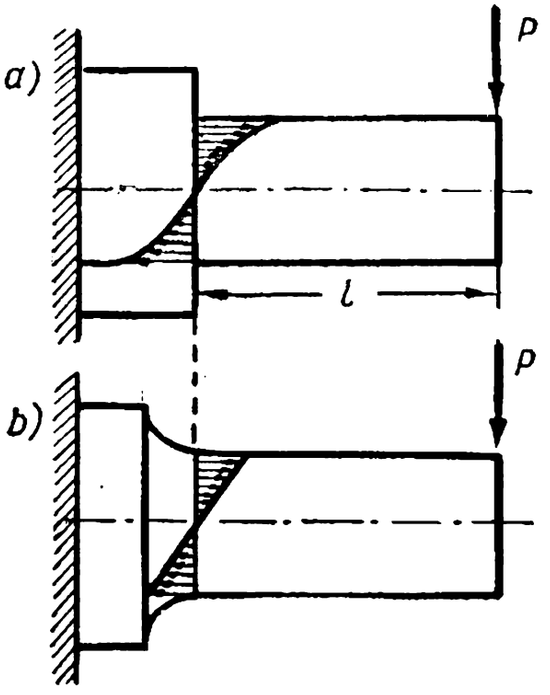


Fig 6.30

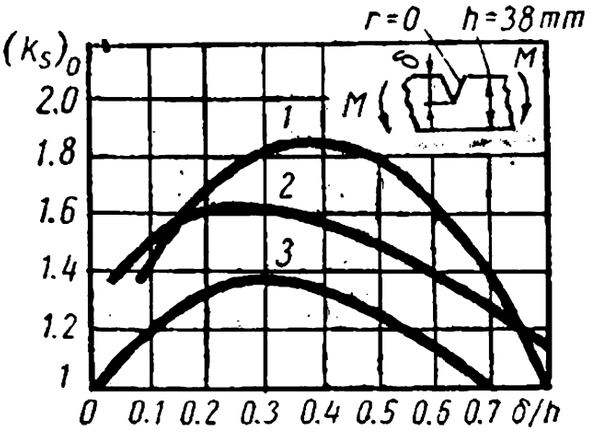


Fig. 6.31

ultimate bending strength is found for a specimen without stress concentration $\sigma_{u.b}$ and for a specimen with stress concentration $\sigma_{u.b.c}$.

The ratio $k_s = \frac{\sigma_{u.b}}{\sigma_{u.b.c}}$ determines the magnitude of the effective stress concentration factor for the specimen. Data concerning the magnitude of k_s are given in handbooks. By way of an example, in Fig. 6.31 data are presented for various materials on the magnitude of $(k_s)_0$ in a rectangular strip having a sharp notch on one side:

(1) curve *1* — aluminium alloy, $\sigma_u = 16$ kg/mm^2;
(2) curve *2* — grey and nickel cast iron;
(3) curve *3* — tungsten cast iron.

Stress concentration factors for members having other dimensions are calculated by multiplying $(k_s)_0$ by correction factors whose values are given in handbooks.

Under repeated loads (in fatigue analysis) stress concentration is taken into account for all materials.

57. Potential Energy of Strain in Bending

In bending, just as in other types of deformation, the work of external forces produces a change in the potential energy of a strained bar.

Let us calculate the potential energy in a case of pure bending. The work done by the external moment can be expressed as one half the product of the moment and the angle of rotation of the section (see Fig. 6.16):

$$dA = \frac{1}{2}\mathfrak{M}\,d\varphi.$$

The work done by the internal bending moments will be of equal magnitude but negative in sign:

$$dU = -\frac{1}{2}M_b\,d\varphi.$$

It can be observed from the drawing that

$$d\varphi = \frac{dz}{\rho}.$$

We have previously obtained the expression for curvature

$$\frac{1}{\rho} = \frac{M_b}{EJ_x},$$

therefore

$$dU = -\frac{1}{2}\frac{M_b^2}{EJ_x}dz.$$

In pure bending M_b=const, and the total work done by the internal forces for a beam of length l will be given by

$$U = -\frac{1}{2}\frac{M_b^2 l}{EJ_x}. \tag{6.20}$$

The potential energy, which is equal to the work done by the internal forces with the sign reversed, will be

$$\Pi = -U = \frac{1}{2}\frac{M_b^2 l}{EJ_x}. \tag{6.21}$$

This formula is analogous in its structure to the expression for potential energy in tension and torsion.

In the general case of bending (in transverse bending) when both bending moments and shearing forces are produced at cross sections of a beam, the potential energy of

strain will be made up of two parts, one corresponding to the work done by the bending moments and the other to the work done by the shearing forces.

The work done by the bending moment varying along the length of a beam will be obtained by integrating the elementary work:

$$U = -\frac{1}{2}\int_0^l \frac{M_b^2}{EJ_x}\,dz. \tag{6.22}$$

The work done by the shearing force is usually very small, as estimations show, compared with the work done by the bending moment and it is neglected, as a rule.

CHAPTER VII

BENDING. DETERMINATION OF DISPLACEMENTS

58. Differential Equation of Elastic Curve

A beam bends under the action of a load. The cross sections of the beam displace perpendicular * to the initially straight axis of the beam and at the same time rotate (Fig. 7.1).

The displacement of the centroid of the cross section in a direction perpendicular to the axis of the beam is called *the deflection* of the beam at a given point (at a given section) and is denoted by the letter y. The angle θ through which the cross section rotates with respect to its original position is called *the angle of rotation of the section.* Bearing in mind that the cross section after rotation is perpendicular to the deflected axis of the beam we conclude that instead of determining the angle of rotation of the section it is possible to determine the angle of equal magnitude between the tangent to the deflected axis at a given point and the original axis of the beam (see Fig. 7.1, where the deflection and the angle of rotation of the section are given for a point A).

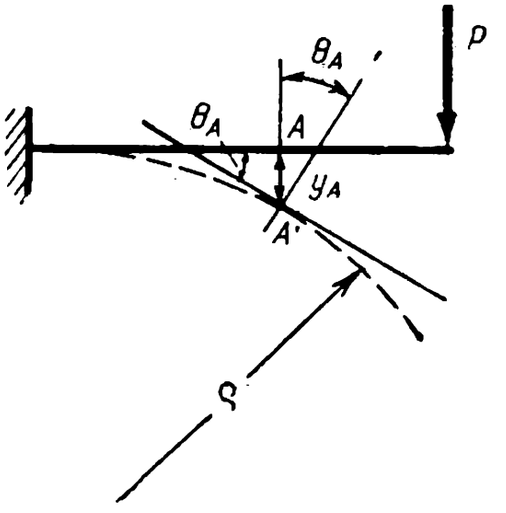


Fig. 7.1

To determine deformations of the beam we shall use the equation relating the curvature of the axis of the beam to

* Strictly speaking, the centroids of the cross sections of the beam displace in bending not only perpendicular to the axis but also parallel to it, but these latter displacements are very small in comparison with the first displacements and can therefore be neglected.

the bending moment and the flexural rigidity of the beam (see Section 52):

$$K=\frac{1}{\rho}=\frac{M_b}{EJ}. \tag{7.1}$$

From mathematics the following formula is known for the curvature of a line:

$$K=\pm\frac{y''}{[1+(y')^2]^{3/2}}. \tag{7.2}$$

Substituting this value of K in formula (7.1) we obtain

$$\pm\frac{y''}{[1+(y')^2]^{3/2}}=\frac{M_b}{EJ}. \tag{7.3}$$

Equation (7.3) represents the exact differential equation of the deflected axis of the beam (the elastic curve). Integration of this non-linear equation presents great difficulties. However, in most practical problems the quantity $(y')^2=\tan^2\theta\approx\theta^2$ may be neglected in comparison with unity since deformations are small. Thus we obtain the approximate differential equation of the deflected axis of the beam:

$$\pm EJy''=M_b \tag{7.4}$$

integration of which offers no difficulty. The choice of the sign is determined by the co-ordinate system adopted. For the co-ordinate axes in the position shown in Fig. 7.2*a* we have the same signs for curvature $K\approx y''$ and for moment M_b. Consequently, in this co-ordinate system equation (7.4) should be used in the form

$$EJy''=M_b. \tag{7.5}$$

For the co-ordinate axes in the position shown in Fig. 7.2*b* we have different signs for y'' and M_b. Consequently, in this case equation (7.4) should be used in the form

$$EJy''=-M_b. \tag{7.6}$$

We shall go on to use the co-ordinate system shown in Fig. 7.2*a* and the differential equation of the deflected axis as given by (7.5).

For calculating angles of rotation $\theta\approx y'$ and deflections y it is necessary to integrate equation (7.5), which can be done by three methods: analytical, semi-graphical and graphical. Let us consider the analytical method.

By integrating equation (7.5) once we obtain the equation for angles of rotation:

$$EJy' = \int M_b\,dz + C, \tag{7.7}$$

where C is the constant of integration. By integrating a second time we obtain the equation of deflections:

$$EJy = \int dz \int M_b\,dz + Cz + D, \tag{7.8}$$

where D is the second constant of integration.

The constants of integration C and D are determined from the supporting conditions of the beam (the boundary

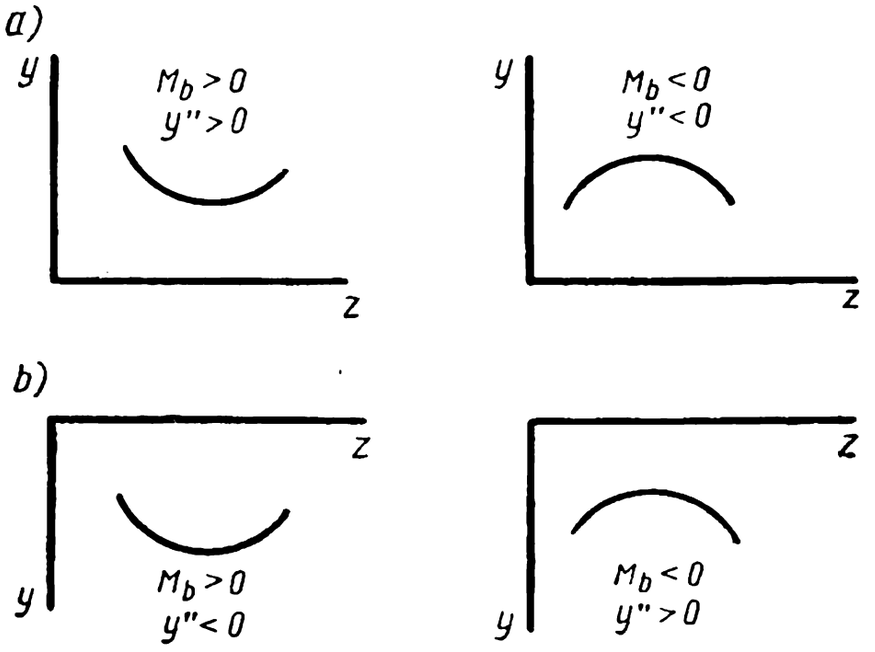


Fig. 7.2

conditions). Thus, for a beam built-in at one end (see Fig. 7.1) both the deflection and the angle of rotation of the section at the built-in end must be equal to zero. For a beam with supported ends the deflection must be zero at the left as well as at the right end.

Having determined the constants of integration the angle of rotation and the deflection of any section can be found from equations (7.7) and (7.8).

In many cases, due to service considerations the maximum deflections of beams are limited by a certain value—*the allowable deflection* [y]. * The magnitude of the allow-

* In special technical and reference literature the magnitude of the allowable sag (maximum deflection) is usually denoted by [f].

able deflection depends on the type of structure or machine. For instance, for crane beams

$$[y] = \left(\frac{1}{600} \div \frac{1}{700}\right) l,$$

where l is the span of the beam.

In mechanical engineering the allowable deflection varies over a rather wide range; depending on the purpose for which the part is designed $[y] = \left(\frac{1}{1000} \div \frac{1}{300}\right) l$. The maximum slope of the sections over the supports must not exceed 0.001 radians.

Example 7.1. Determine y_{max} and θ_{max} for a cantilever loaded by a concentrated force at its end (Fig. 7.3).

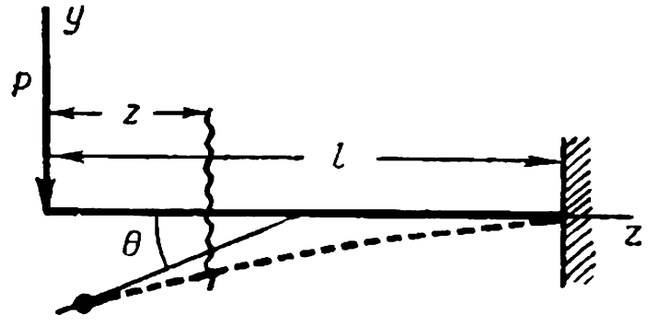


Fig. 7.3

Solution. Place the origin of co-ordinates at the left end of the beam. The bending moment at a section having abscissa z will be determined as the moment of the external forces acting between the section in question and the origin:

$$M_z = -Pz.$$

Therefore, $EJy'' = -Pz$.

The first integration gives:

$$EJy' = -\frac{Pz^2}{2} + C.$$

The second integration gives:

$$EJy = -\frac{Pz^3}{6} + Cz + D.$$

For determining C and D we have the following conditions:

(1) at $z=l$ $y=0$;

(2) at $z=l$ $\theta = y' = 0$.

From the second condition we obtain $C = \frac{Pl^2}{2}$.

From the first condition we obtain $0 = -\frac{Pl^3}{6} + \frac{Pl^3}{2} + D$, whence

$$D = -\frac{Pl^3}{3}.$$

Now we can determine y_{max} and θ_{max}. It is quite evident that y_{max} and θ_{max} will occur at $z=0$. Assuming $z=0$ in formulas we obtain

$$y'_{max} = \theta_{max} = \frac{Pl^2}{2EJ},$$

$$y_{max} = -\frac{Pl^3}{3EJ}. \tag{7.9}$$

The positive value of the angle of rotation θ indicates that the section rotates counter-clockwise.

The negative value of y indicates that the centroid of the section displaces downward, i.e., in the direction of negative values of ordinates y.

Note that

$$C = EJ\theta_0 \tag{7.10}$$

and

$$D = EJy_0, \tag{7.11}$$

where y_0 is the deflection at the origin and θ_0 is the angle of rotation at the origin. Relations (7.10) and (7.11) will always be valid for beams having one region of loading provided bending moments are calculated by considering forces lying between the section and the origin. The same holds for beams with any number of regions of loading if certain artificial procedures are employed for integrating the differential equation of the elastic curve. The above-mentioned methods will be used below in the derivation of the so-called universal equation of the deflected axis of the beam.

59. Determination of Displacements When There Are Several Regions of Loading. Universal Formula

If a beam is subjected to several loads (Fig. 7.4), the law of variation of bending moments in different regions will be expressed by different equations. The differential

equation of the deflected axis must be derived for each region.

The number of constants of integration will be twice the number of regions. To determine these constants it is always possible to derive a sufficient number of equations by using the conditions at the supports of a beam and the conditions at the boundaries between the adjacent regions where deflections and angles of rotation are equal. If no special procedures are employed, the problem of determining arbitrary constants when the number of regions is great will involve much work. By applying special procedures the number of constants of integration can be reduced to two, regardless of the number of regions.

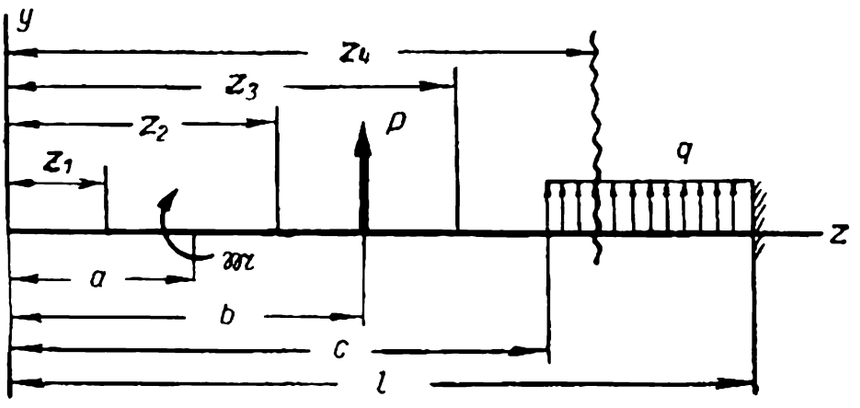


Fig. 7.4

These procedures are as follows:

1. The equation for the bending moment at a section will always be derived by calculating the moment of external forces lying between the section and the origin.

2. The equation will be integrated without opening the brackets.

3. To obtain a more convenient final formula for the external moment $\mathfrak{M}$ introduce a factor $(z-a)^0$, equal to unity. Let us derive and integrate the differential equation of the deflection curve for each of the four regions of the beam:

Region I	Region II
$M_z = 0$	$M_z = \mathfrak{M} = \mathfrak{M}(z_1 - a)^0$
$EJy_1'' = 0$	$EJy_2'' = \mathfrak{M}(z_2 - a)^0$
$EJy_1' = C_1$	$EJy_2' = \mathfrak{M}(z_2 - a) + C_2$
$EJy_1 = C_1 z_1 + D_1;$	$EJy_2 = \frac{\mathfrak{M}(z_2 - a)^2}{2} + C_2 z_2 + D_2;$

Region III

$$M_z = \mathfrak{M}(z_3 - a)^0 + P(z_3 - b)$$

$$EJy_3'' = \mathfrak{M}(z_3 - a)^0 + P(z_3 - b)$$

$$EJy_3' = \mathfrak{M}(z_3 - a) + \frac{P(z_3 - b)^2}{2} + C_3$$

$$EJy_3 = \frac{\mathfrak{M}(z_3 - a)^2}{2} + \frac{P(z_3 - b)^3}{6} + C_3 z_3 + D_3;$$

Region IV

$$M_z = \mathfrak{M}(z_4 - z)^0 + P(z_4 - b) + \frac{q(z_4 - c)^2}{2}$$

$$EJy_4'' = \mathfrak{M}(z_4 - a)^0 + P(z_4 - b) + \frac{q(z_4 - c)^2}{2}$$

$$EJy_4' = \mathfrak{M}(z_4 - a) + \frac{P(z_4 - b)^2}{2} + \frac{q(z_4 - c)^3}{6} + C_4$$

$$EJy_4 = \frac{\mathfrak{M}(z_4 - a)^2}{2} + \frac{P(z_4 - b)^3}{6} + \frac{q(z_4 - c)^4}{24} + C_4 z_4 + D_4.$$

For determining the eight constants of integration we have the following conditions:

(1) at $z_1 = z_2 = a$ $\quad y_1 = y_2$

(2) at $z_1 = z_2 = a$ $\quad y_1' = y_2'$

(3) at $z_2 = z_3 = b$ $\quad y_2 = y_3$

(4) at $z_2 = z_3 = b$ $\quad y_2' = y_3'$

(5) at $z_3 = z_4 = c$ $\quad y_3 = y_4$

(6) at $z_3 = z_4 = c$ $\quad y_3' = y_4'$

(7) at $z_4 = l$ $\quad y_4 = 0$

(8) at $z_4 = l$ $\quad y_4' = 0.$

By using these conditions we obtain

$$C_1 = C_2 = C_3 = C_4 = C,$$

$$D_1 = D_2 = D_3 = D_4 = D.$$

Thus, in place of eight constants there remain only two, which are determined from the seventh and eighth conditions. As has already been pointed out, the arbitrary con-

stant C represents the angle of rotation at the origin multiplied by the flexural rigidity of the beam, i.e.,

$$C = EJ\theta_0,$$

and the constant D represents the deflection at the origin multiplied by EJ, i.e.,

$$D = EJy_0.$$

Analysing the expressions for the deflection of the beam we see that the following type of equation is of the most general form:

$$EJy = D + Cz + \frac{\mathfrak{M}(z-a)^2}{2} + \frac{P(z-b)^3}{6} + \frac{q(z-c)^4}{24}.$$

Bearing in mind that $D = EJy_0$ and $C = EJ\theta_0$ and considering a case when several moments and forces are applied we obtain the so-called *universal formula*, which is unwieldy in appearance but convenient in application, especially under combined loading:

$$EJy = EJy_0 + EJ\theta_0 z + \Sigma \frac{\mathfrak{M}(z-a)^2}{2} + \\ + \Sigma \frac{P(z-b)^3}{6} + \Sigma \frac{q(z-c)^4}{24}. \qquad (7.12)$$

Here $\mathfrak{M}$, P and q are the external forces and moments (including the reactions of the supports) lying between the section in question and the origin, y_0 and θ_0 are the deflection and the angle of rotation at the origin. In a similar way or by differentiating formula (7.12) we obtain the universal formula for the angle of rotation:

$$EJy' = EJ\theta_0 + \Sigma \mathfrak{M}(z-a) + \Sigma \frac{P(z-b)^2}{2} + \Sigma \frac{q(z-c)^3}{6}. \qquad (7.13)$$

Here a, b, c are the distances from the origin to the points of application of the moment, the concentrated force and the beginning of the portion subjected to a uniformly distributed load, respectively. If the direction of forces and moments is opposite to that accepted in the derivation of the formula, they must be taken as negative in sign.

It should be noted that the last term in these formulas is valid only if the distributed load does not break off before the section where y or θ are determined. If the load does break off, it should be continued up to the section

in question, and at the same time a load equal in magnitude but of the opposite sense should be added (Fig. 7.5).

The shortcoming of the universal formulas is that they cannot be used directly to determine displacements in beams having variable flexural rigidity EJ in different portions. The foregoing formulas are not applicable at all to

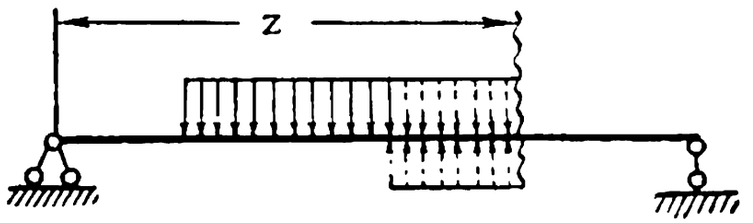


Fig. 7.5

beams whose cross sections vary continuously along the length. In these cases use should be made of the general method of determining displacements, the Mohr-Vereshchagin method (see further).

60. Examples of Determining Displacements in Bending by Use of the Universal Formula

Example 7.2. Determine the maximum deflection and angle of rotation for a cantilever subjected to a uniformly distributed load (Fig. 7.6).

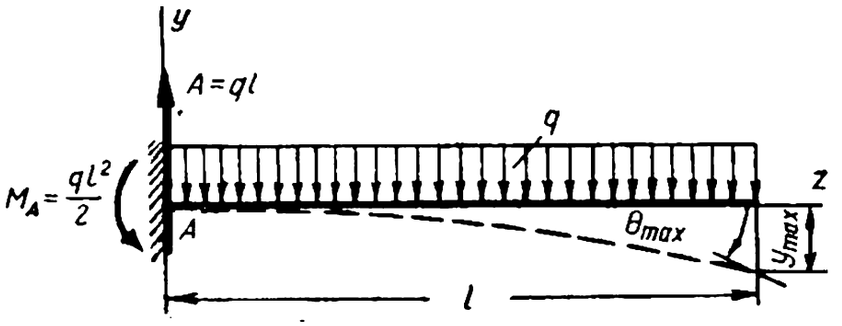


Fig. 7.6

Solution. It is better to place the origin at the left end of the beam since then $y_0=0$ and $\theta_0=0$.

Consequently, y_{max} and θ_{max} can readily be calculated according to the universal formula; they occur, as can be observed from the drawing, at section $z=l$.

When using the universal formula it is necessary, as stated previously, to take forces and moments lying be-

tween the section in question and the origin. For that determine first the reactive moment and reactive force at the built-in end.

The reactive force $A=ql$ acts upward. It will enter in the universal formula with a plus sign. The reactive moment $M_A=\frac{ql^2}{2}$ is counter-clockwise. It will enter in the universal formula with a minus sign.

The distances from the origin to the reactive moment and the beginning of the uniformly distributed load are zero, i.e.,

$$a=b=c=0.$$

Determine y at $z=l$, i.e., $y_{\max}$:

$$EJy_{\max}=-\frac{ql^2}{2}\frac{(l-0)^2}{2}+\frac{ql(l-0)^3}{6}-\frac{q(l-0)^4}{24}=-\frac{ql^4}{8}.$$

Hence

$$y_{\max}=-\frac{ql^4}{8EJ}. \tag{7.14}$$

From the universal formula for angles of rotation we obtain

$$EJy'_{\max}=-\frac{ql^2}{2}(l-0)+\frac{ql(l-0)^2}{2}-\frac{q(l-0)^3}{6}=-\frac{ql^3}{6}.$$

Hence

$$y'_{\max}=\theta_{\max}=-\frac{ql^3}{6EJ}.$$

Example 7.3. Determine $y_{\max}$ and $\theta_{\max}$ for the beam shown in Fig. 7.7.

Solution. From symmetry the reactions are equal to

$$A=B=\frac{ql}{2}.$$

Place the origin at the left support. Then $y_0=0$.

To determine θ_0 make use of the condition that $y=0$ at $z=l$. We obtain

$$EJy_{z=l}=EJ\theta_0 l+\frac{ql}{2}\frac{(l-0)^3}{6}-\frac{q(l-0)^4}{24}=0,$$

whence

$$\theta_0=-\frac{ql^3}{24EJ}=\theta_A.$$

Obviously, $\theta_A = -\theta_B$. Note that the sections over the supports have the maximum angles of rotation, i.e., $\theta_{max} = |\theta_A| = |\theta_B|$.

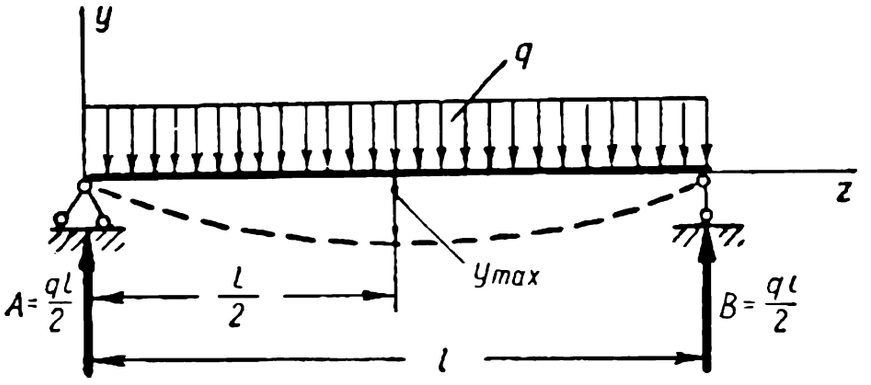


Fig. 7.7

The maximum deflection occurs at the mid-span of the beam:

$$EJy_{z=\frac{l}{2}} = -\frac{ql^3}{24}\frac{l}{2} + \frac{\frac{ql}{2}\left(\frac{l}{2}-0\right)^3}{6} - \frac{q\left(\frac{l}{2}-0\right)^4}{24} = -\frac{5ql^4}{384}.$$

Consequently,

$$y_{max} = -\frac{5ql^4}{384EJ}. \qquad (7.15)$$

Example 7.4. Determine the maximum deflection and the angle of rotation at the supports for a beam loaded by a concentrated force at the mid-span (Fig. 7.8).

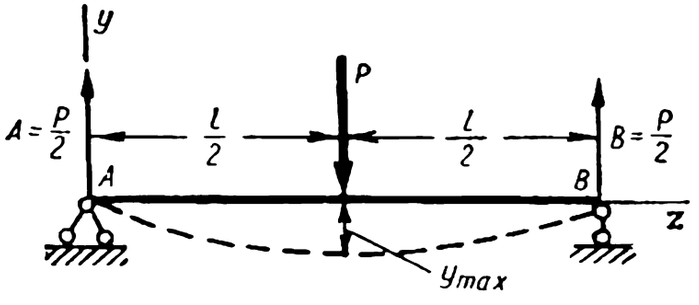


Fig. 7.8

Solution. The reactions are $\frac{P}{2}$ each and act upwards.

Place the origin at the left end, then $y_0 = 0$. To determine θ_0 make use of the condition that at $z = l$ the deflec-

tion is zero: $y=0$. *

$$EJy_{z=l}=EJ\theta_0 l+\frac{P}{2}\frac{(l-0)^3}{6}-\frac{P\left(l-\frac{l}{2}\right)^3}{6}=0,$$

whence

$$EJ\theta_0=-\frac{Pl^2}{16}.$$

Consequently,

$$\theta_0=\theta_A=\theta_{\max}=-\frac{Pl^2}{16EJ}.$$

Due to symmetry the angle of rotation at the right support is

$$\theta_B=-\theta_A=\frac{Pl^2}{16EJ}.$$

The maximum deflection is $y_{\max}=y_{z=\frac{l}{2}}$,

therefore,

$$EJy_{\max}=-\frac{Pl^2}{16}\times\frac{l}{2}+\frac{P}{2}\frac{\left(\frac{l}{2}\right)^3}{6}=-\frac{Pl^3}{48}.$$

Finally

$$y_{\max}=-\frac{Pl^3}{48EJ}. \qquad (7.16)$$

Example 7.5. Determine the deflections at points D and C and the angle of rotation at point B of the beam represented in Fig. 7.9. The moment of inertia of the cross section of the beam is $J=15{,}760$ cm^4.

$$E=2\times10^6 \text{ kg/cm}^2.$$

Solution. Determine the reactions of the supports:

(1) $\Sigma M_B=0$, $\quad A\times4+4\times3-8-4\times1-4\times2=0$,

$$A=2 \text{ t};$$

(2) $\Sigma Y=0$, $\quad -2-4+B-4-4=0$, $\quad B=14$ t.

Place the origin at the left support. Then $y_0=0$. The angle θ_0 is determined from the condition that $y=0$ at $z=4$ m.

---

* In this case θ_0 could also have been determined from the condition that the tangent to the elastic curve at the mid-span is horizontal, i.e., $y'_{z=\frac{l}{2}}=0$.

According to the universal formula for deflections we have at $z=4$ m

$$EJy_{z=4} = EJ\theta_0 \times 4 + \frac{8(4-2)^2}{2} - \frac{2(4-0)^3}{6} - \frac{2(4-0)^4}{24} + \frac{2(4-2)^4}{24} = 0.$$

Since the distributed load breaks off at point D we continue it, in accord with the above, up to the end, introducing a compensating load of the opposite sense in portion

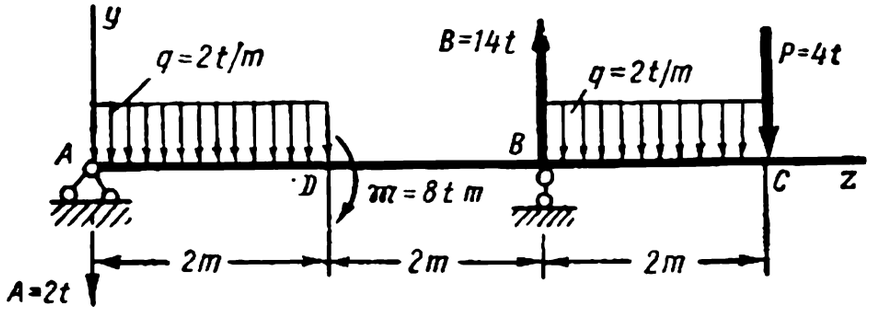


Fig. 7.9

DB. It is the last term that allows for the distributed load acting upwards (the above transformations of the load are not indicated in the drawing).

Making calculations we obtain

$$\theta_0 = \theta_A = \frac{6.33}{EJ}.$$

Determine the deflection at point C ($z=6$ m):

$$EJy_C = 6.33 \times 6 + \frac{8(6-2)^2}{2} - \frac{2(6-0)^3}{6} + \frac{14(6-4)^3}{6} - \frac{2(6-0)^4}{24} + \frac{2(6-2)^4}{24} - \frac{2(6-4)^4}{24} = -39.3 \text{ tm}^3 = 39.3 \times 10^9 \text{ kg cm}^3.$$

Determine the deflection at point D:

$$EJy_D = 6.33 \times 2 - \frac{2(2-0)^3}{6} - \frac{2(2-0)^4}{24} = 8.67 \text{ tm}^3 = 8.67 \times 10^9 \text{ kg cm}^3.$$

We obtain finally

$$y_C = \frac{-39.3 \times 10^9}{EJ} = \frac{-39.3 \times 10^9}{2 \times 10^6 \times 15{,}760} = -1.26 \text{ cm},$$

$$y_D = \frac{8.67 \times 10^9}{EJ} = \frac{8.67 \times 10^9}{2 \times 10^6 \times 15{,}760} = 0.275 \text{ cm}.$$

Determine the angle of rotation at point B. According to the universal formula for angles of rotation we have at $z=4$ m

$$EJy'_B = 6.33 + 8(4-2) - \frac{2\times 4^2}{2} - \frac{2\times 4^3}{6} + \frac{2(4-2)^3}{6} =$$

$$= -12.3 \text{ tm}^2 = -12.3\times 10^7 \text{ kg cm}^2,$$

whence

$$\theta_B = \frac{-12\times 3\times 10^7}{EJ} = \frac{-12.3\times 10^7}{2\times 10^6\times 15{,}760} = -39\times 10^{-4} \text{ radians.}$$

61. Reciprocal Theorem

Let us now prove a theorem having important applications, namely the reciprocal theorem or Betti's theorem after the Italian scientist who was the first to publish it. For that consider a linearly deformed system in two different states corresponding to two different loads (Fig. 7.10). For simplicity of calculations consider a simple beam subjected in both states to the simplest load (to one concentrated force). The load, internal forces and deformations corresponding to these two states are marked by subscripts 1 and 2.

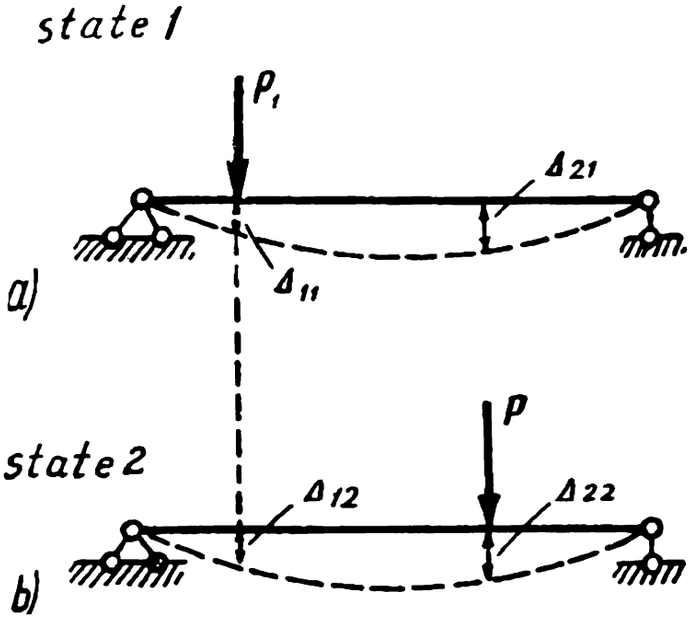


Fig. 7.10

Fig. 7.10*a* shows the first state of the system and Fig. 7.10*b* the second one. The letter Δ_{11} denotes the displacement in the direction of the load of the first state due to this load.

The letter Δ_{21} denotes the displacement in the direction of the load of the second state caused by the action of the load of the first state.

The notation for displacements of the second state is shown in Fig. 7.10*b*. The displacements containing in their symbols two identical subscripts, as for instance, Δ_{11}, Δ_{22},

are called principal, and such displacements as Δ_{12}, Δ_{21}, etc., are called subsidiary. Let us now prove the reciprocal theorem, namely, that *the work done by the external or internal forces of the first state over the displacements of the second state is equal to the work done by the forces of the second state over the displacements caused by the forces of the first state.*

To prove the theorem let us load the beam with forces P_1 and P_2, applying them in different order.

1. First apply load P_1 and then apply force P_2 to the deformed beam (Fig. 7.11*a*).

Compute the work done by the external forces. The work done by force P_1 over its own displacement Δ_{11} caused by this force will be

$$A_{11}=\frac{P_1\Delta_{11}}{2}$$

[see formula (2.44)].

The work done by force P_2 over its own displacement Δ_{22} will be

$$A_{22}=\frac{P_2\Delta_{22}}{2}.$$

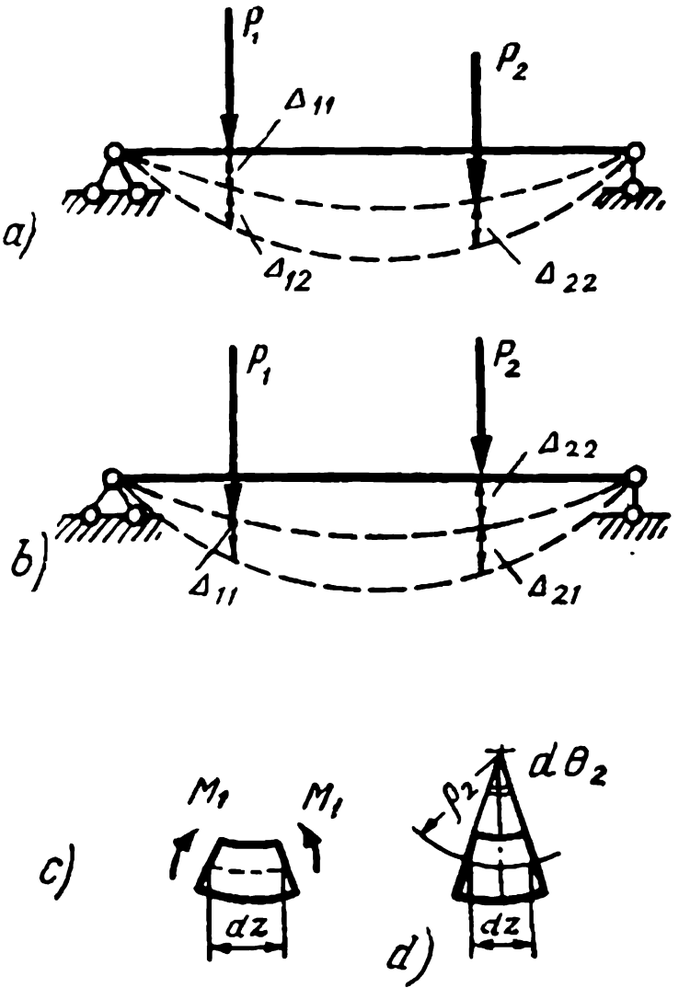


Fig. 7.11

The additional work done by force P_1 over the displacement Δ_{12} caused by force P_2 is

$$A_{12}=P_1\Delta_{12}.$$

Note that in computing A_{12} the factor $\frac{1}{2}$ is omitted because force P_1 does the work over the displacement Δ_{12} having its full value. The work done by a force over the displacement caused by another force is called *the virtual work*. The total work done by the external forces in the first type of loading is

$$A_I=A_{11}+A_{22}+A_{12}.$$

2. Let us now load the beam in the other order: first apply force P_2 and then force P_1 (Fig. 7.11*b*). The work done by force P_2 over its own displacement Δ_{22} will be

$$A_{22} = \frac{P_2 \Delta_{22}}{2}.$$

The work done by force P_1 over its own displacement Δ_{11} will be

$$A_{11} = \frac{P_1 \Delta_{11}}{2}.$$

The work done by force P_2 over the displacement Δ_{21} will be

$$A_{21} = P_2 \Delta_{21}.$$

The total work in the second type of loading is

$$A_{II} = A_{22} + A_{11} + A_{21}.$$

The work done by the forces, however, does not depend on the order of their application. Therefore,

$$A_I = A_{II},$$

whence

$$A_{12} = A_{21}. \tag{7.17}$$

Or for the case being considered

$$P_1 \Delta_{12} = P_2 \Delta_{21}. \tag{7.18}$$

This proves the above formulated reciprocal theorem concerning the virtual work done by external forces. We have proved it for concentrated external loads. However, the theorem is valid for any external load as well: concentrated, distributed, and external moments. Just one thing should be kept in mind—the work done by moments is then computed over angular displacements rather than linear ones.

In a similar way the reciprocal theorem can also be proved in respect of the virtual work done by internal forces:

$$U_{12} = U_{21}. \tag{7.19}$$

The virtual work, for instance U_{12}, for an element of the beam of length dz will be (Fig. 7.11 *c* and *d*): $dU_{12} = M_1 d\theta_2$.

It can be observed from the drawing that

$$d\theta_2 = \frac{dz}{\rho_2}.$$

But $\frac{1}{\rho_2} = \frac{M_2}{EJ}$ (see formula 6.7).

Hence

$$d\theta_2 = \frac{M_2\, dz}{EJ}$$

and therefore

$$dU_{12} = \frac{M_1 M_2\, dz}{EJ}.$$

The virtual work U_{12} for the entire beam of length l is

$$U_{12} = \int_0^l \frac{M_1 M_2}{EJ}\, dz. \tag{7.20}$$

Here M_1 and M_2 are the current values of the bending moments in the first and in the second state.

Note that here just as in computing the virtual work done by the external forces the factor $^1/_2$ is not present.

In a similar way it may be shown that the work done by the internal forces of the second state over the displacements caused by the internal forces of the first state can be calculated by the formula

$$U_{21} = \int_0^l \frac{M_2 M_1}{EJ}\, dz. \tag{7.21}$$

Comparing expressions (7.20) and (7.21), we see that in fact

$$U_{12} = U_{21}$$

as

$$\int_0^l \frac{M_1 M_2}{EJ}\, dz = \int_0^l \frac{M_2 M_1}{EJ}\, dz.$$

This proves the reciprocity of the virtual work done by the internal forces.

By using the law of conservation of energy it can be shown that *the virtual work done by the external forces is*

equal in magnitude to the virtual work done by the internal forces:

$$A_{12} = U_{12}$$

or

$$A_{21} = U_{21}.$$

On the basis of the above it also follows that

$$A_{12} = A_{21} = U_{12} = U_{21}. \tag{7.22}$$

These relations will be used later to justify Mohr's method of determining displacements.

As a particular case, another important theorem follows from the reciprocal theorem for the work—*the reciprocal theorem of displacements* (Maxwell's theorem).

Assuming $P_1 = P_2 = 1$ we obtain from Betti's theorem [from formula (7.18)]

$$\delta_{12} = \delta_{21}, \tag{7.23}$$

i.e., *the displacement of the point of application of the first unit force in its direction caused by the second unit force is equal to the displacement of the point of application of the second unit force in the direction of the latter caused by the action of the first unit force.* Note that displacements caused by forces equal to unity (unit forces) are denoted by δ_{11}, δ_{12}, etc., in place of the symbols Δ_{11}, Δ_{12}, etc., used for displacements due to forces of any magnitude.

62. Mohr's Method of Determining Displacements

Consider now the general method of determining displacements suitable for any linearly deformed system under any loading. This method was proposed by the eminent German scientist Mohr.

Let it be required, for instance, to determine the vertical displacement of point A in the beam represented in Fig. 7.12*a*. Denote the given state of loading by k. Choose a subsidiary (or fictitious) state of the same beam with the unit force acting at the same point and in the direction of the unknown deflection. Denote the subsidiary (fictitious) state by i (Fig. 7.12*b*).

Let us compute the virtual work done by the external and internal forces in the fictitious state over the displacements caused by the action of the forces in the state of loading.

The work done by the external forces will be equal to the product of the unit force and the unknown displacement y_A

$$A_{ik} = 1 \times y_A,$$

and the work done by the internal forces will be given by the integral

$$U_{ik} = U_{ki} = \int_0^l \frac{M_i M_k \, dz}{EJ}.$$

But

$$A_{ik} = U_{ik} \text{ (see formula 7.22).}$$

Hence we obtain Mohr's remarkable formula (Mohr's integral)

$$y_A = \int_0^l \frac{M_i M_k \, dz}{EJ}, \tag{7.24}$$

which makes it possible to determine the displacement of any linearly deformable system (i.e., a system obeying Hooke's law) at any point and in any direction.

In this formula the product $M_i M_k$ under the integral sign is positive if both bending moments have the same sign.

If it were necessary to determine the angular displacement at point A, the moment equal to unity should be applied at point A in the fictitious state.

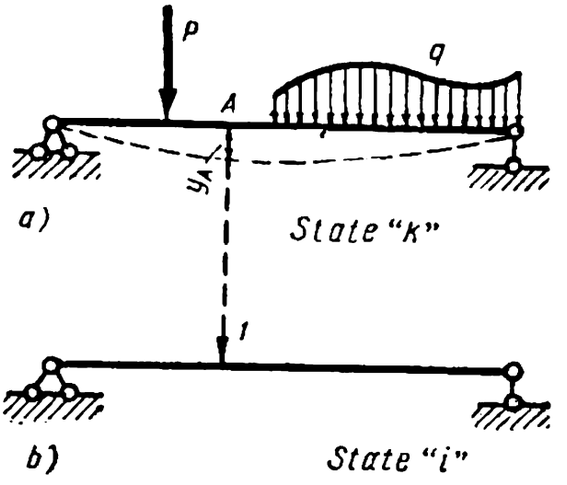


Fig. 7.12

Denoting any displacement (linear or angular) by the letter Δ, Mohr's formula (integral) will be written in the form

$$\Delta = \int_0^l \frac{M_i M_k \, dz}{EJ}. \tag{7.25}$$

The analytical expression of M_i and M_k may be different in different portions of a beam or an elastic system

in general. A more general formula should therefore be used in place of formula (7.25):

$$\Delta = \Sigma \int_0^l \frac{M_i M_k \, dz}{EJ}. \tag{7.26}$$

If bars of a system work in tension (compression) and not in bending, as for instance in trusses, Mohr's formula will be

$$\Delta = \Sigma \int_0^l \frac{N_i N_k \, dz}{EF}. \tag{7.27}$$

In this formula the product $N_i N_k$ is positive if both forces are tensile or both compressive. If bars are under combined bending and tension (compression), in most cases, as com-

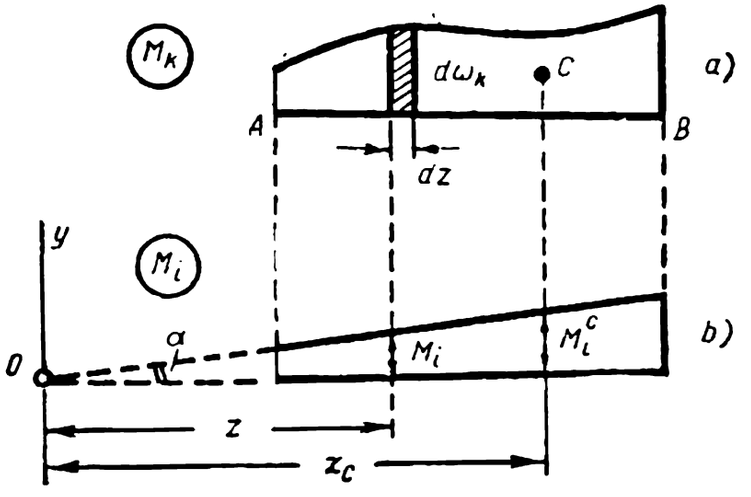


Fig. 7.13

parative calculations show, displacements can be determined by taking into account bending moments only, the effect of longitudinal forces being very slight.

For the same reasons, in ordinary cases the effect of shearing forces can be neglected.

Instead of computing Mohr's integral directly use can be made of the semi-graphical procedure called "method of multiplying the diagrams", or Vereshchagin's rule.

Consider two bending moment diagrams, one of which M_k has an arbitrary configuration, and the other M_i is a straight line (Fig. 7.13).

The cross section of the bar in portion AB will be considered constant. In this case

$$\int_0^l \frac{M_i M_k\, dz}{EJ} = \frac{1}{EJ} \int_0^l M_i M_k\, dz. \tag{7.28}$$

The quantity $M_k dz$ represents an elementary area $d\omega_k$ of the M_k diagram (shaded in the figure).
Thus

$$\int_A^B M_i M_k\, dz = \int_A^B M_i\, d\omega_k. \tag{7.29}$$

But

$$M_i = z \tan \alpha. \tag{7.30}$$

Therefore

$$\int_A^B M_i\, d\omega_k = \tan \alpha \int_A^B z d\omega_k. \tag{7.31}$$

But $\int_A^B z\, d\omega_k$ represents the statical moment of the area of the M_k diagram with respect to an axis y through point O which is equal, obviously, to $\omega_k z_c$ where ω_k is the area of the bending moment diagram and z_c is the distance from the y-axis to the centroid of the M_k diagram.

From the drawing it can be observed that

$$z_c = \frac{M_i^c}{\tan \alpha}, \tag{7.32}$$

where M_i^c is the ordinate to the M_i diagram located under the centroid of the M_k diagram (under point C).

Therefore,

$$\int M_i\, d\omega_k = \tan \alpha \int z d\omega_k = \tan \alpha\ \ \omega_k z_c = \omega_k M_i^c, \tag{7.33}$$

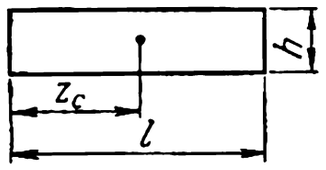
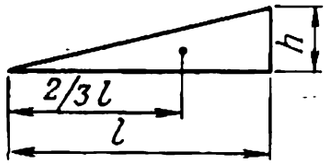
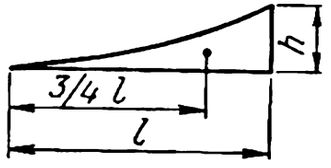
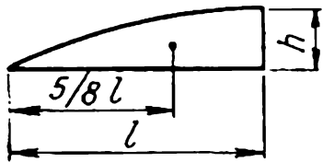
i.e., *the integral in question is equal to the product of the area of the M_k diagram (of any configuration) and the ordinate through its centroid to the straight line diagram M_i^c.* The value of the quantity $\omega_k M_i^c$ is considered positive if both diagrams are located on one side of the bar; otherwise it is negative.

It should be remembered that the ordinate M_i^c should always be taken in the straight line diagram. In a particular case when both diagrams are straight, the area of either of them may be multiplied by the corresponding ordinate to the other.

When ordinary loads are applied such as external concentrated moments, concentrated forces and uniformly distributed loads any moment diagram can be broken into a series of simpler diagrams: rectangles, triangles, square parabolas.

Table 7.1 presents the values of the areas of these diagrams and the co-ordinates of their centroids.

Table 7.1

Diagram	Area of diagram, ω	Distance to centroid, z_c
	hl	$\frac{1}{2}l$
	$\frac{1}{2}hl$	$\frac{2}{3}l$
	$\frac{1}{3}hl$	$\frac{3}{4}l$
	$\frac{2}{3}hl$	$\frac{5}{8}l$

To speed up the calculations use can be made of available tables for multiplying diagrams (Table 7.2).

This table gives the results of multiplying diagrams in the cells at the intersection of the corresponding diagrams.

When breaking a compound diagram into "elementary diagrams" represented in Tables 7.1 and 7.2 it should be borne in mind that the parabolic diagrams have been obtained with a distributed load acting alone.

In cases where curvilinear portions in a compound diagram result from a simultaneous action of concentrated moments, forces and uniformly distributed loads, to avoid

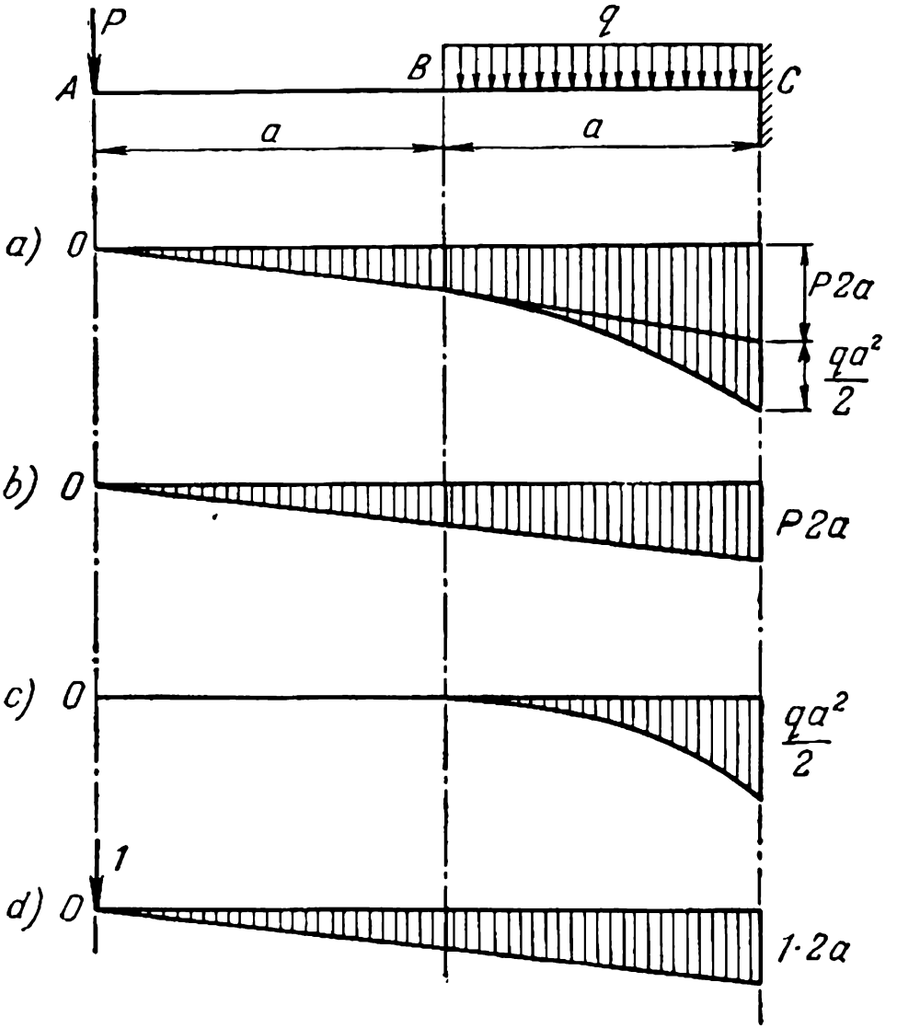


Fig. 7.14

errors it is necessary first to "separate" the compound diagram, i.e., to break it into a series of independent diagrams due to the action of concentrated moments, forces and uniformly distributed loads.

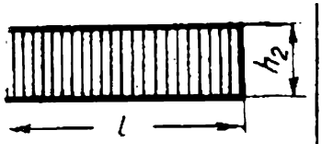
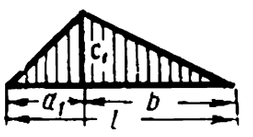
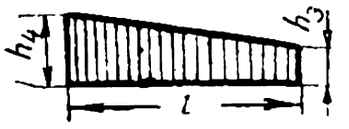
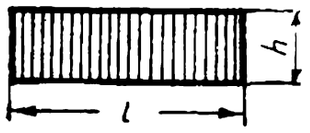
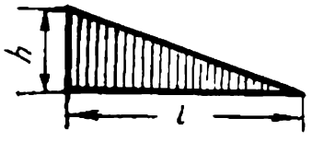
Let it be required, for instance, to determine the vertical displacement of the left end of a beam (Fig. 7.14).

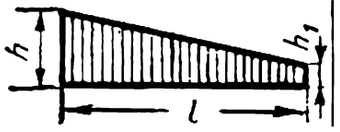
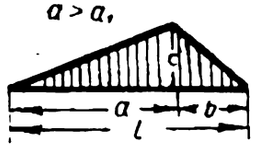
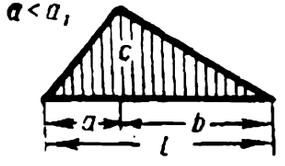
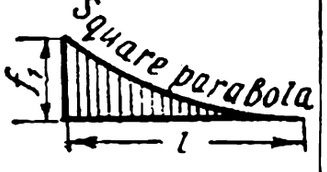
The total diagram due to the load is represented in Fig. 7.14*a*.

The diagram due to the action of the unit force at point *A* is represented in Fig. 7.14*d*.

Table 7.2

Table of Certain Integrals $\int M_i M_k \, dz$

M_k \ M_i				
	hh_2l	$\frac{hh_3l}{2}$	$\frac{c_1l_1h}{2}$	$\frac{h(h_4+h_5)l}{2}$
	$\frac{hh_2l}{2}$	$\frac{hh_3l}{3}$	$\frac{h(l+b_1)c_1}{6}$	$\frac{hl}{6}(2h_4+h_5)$
	$\frac{hh_2l}{2}$	$\frac{hh_3l}{6}$	$\frac{h(a_1+l)c_1}{6}$	$\frac{hl}{6}(h_4+2h_5)$

	$\frac{h_2(h+h_1)l}{2}$	$\frac{h_3 l}{6}(2h+h_1)$	$\frac{c_1 h}{6}[h(l+b_1)+$ $+h_1(l+a_1)]$	$\frac{l}{6}[h(2h_4+h_5)+h_1(2h_5+h_4)]$
	$\frac{ch_2 l}{2}$	$\frac{ch_3(l+b)}{6}$	$\frac{cc_1 l}{6}\left[2-\frac{(b_1-b_2)^2}{b_1 a}\right]$	$\frac{c}{6}[h(l+b_1)+h_1(l+a_1)]$
	$\frac{ch_2 l}{2}$	$\frac{ch_3(l+b)}{6}$	$\frac{cc_1 l}{6}\left[2-\frac{(a_1-a)^2}{a_1 b}\right]$	$\frac{c}{6}[h(l+b_1)+h_1(l+a_1)]$
	$\frac{2flh_2}{3}$	$\frac{flh_3}{3}$	$\frac{c_1 f_1}{3l}(l^2+a_1 b_1)$	$\frac{fl}{3}(h_4+h_5)$
	$\frac{f_1 l h_2}{3}$	$\frac{flh_3}{4}$	$\frac{c_1 f_1}{12}\left[l+b_1+\frac{b_1^2}{l}\right]$	$\frac{f_1 l}{12}[3h_4+h_5)$

To determine the vertical displacement at point A it is necessary to multiply the diagram due to the load by the diagram due to the unit force.

We note however that the curvilinear diagram in the portion BC of the total diagram results not only from the action of the uniformly distributed load but also from the action of the concentrated force P. In consequence the parabolic diagram in the portion BC is no longer one of the elementary diagrams represented in Tables 7.1 and 7.2 but is essentially a compound diagram for which the data of these tables are not valid.

It is therefore necessary to separate the compound diagram in Fig. 7.14*a* into elementary diagrams represented in Figs. 7.14*b* and 7.14*c*.

The diagram in Fig. 7.14*b* results from the action of the concentrated force only, and the diagram in Fig. 7.14*c* results from the action of the uniformly distributed load only.

We can now multiply the diagrams by using Tables 7.1 and 7.2.

To do this it is necessary to multiply the triangular diagram in Fig. 7.14*b* by the triangular diagram in Fig. 7.14*d* and to add to this the result of multiplying the parabolic diagram in Fig. 7.14*c* by the trapezoidal diagram of the portion BC in Fig. 7.14*d* since the ordinates of the diagram in Fig. 7.14*c* are zero in the portion AB.

Vereshchagin's rule for multiplying diagrams is not applicable to bars of varying cross section since the quantity EJ can no longer be put before the integral sign. In this case EJ should be expressed as a function of the abscissa of a cross section and after that Mohr's integral (7.25) should be evaluated.

When the flexural rigidity of a bar changes in steps integration (or multiplication of diagrams) is carried out for each portion separately and then the results are summed up.

Example 7.6. Determine the vertical and angular displacement of a cantilever at the point of application of the load (Fig. 7.15).

Solution. Construct the bending moment diagram for the state of loading (Fig. 7.15*a*).

To determine the vertical displacement choose a fictitious state of the beam with the unit load at the point of application of the load.

Construct the diagram of the bending moment due to this force (Fig. 7.15*b*). Determine the vertical displacement by Mohr's method:

$$y = \int_0^l \frac{M_i M_P \, dz}{EJ}.$$

The value of the bending moment produced by the load is

$$M_P = -Pz.$$

The value of the bending moment produced by the unit force is

$$M_i = -1 \times z.$$

Substitute these values of M_P and M_i under the integral sign and integrate:

$$y = \frac{1}{EJ} \int_0^l (-Pz)(-z)\, dz = \frac{Pl^3}{3EJ}.$$

The same result was obtained previously by another method.

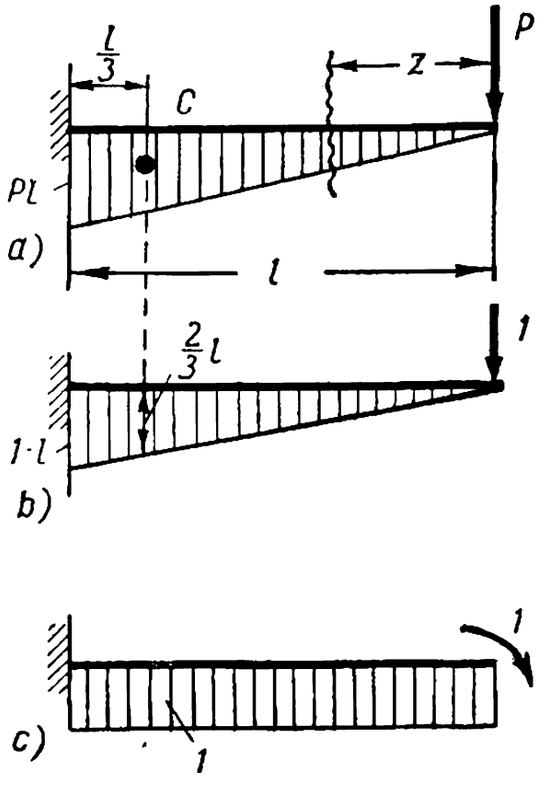


Fig. 7.15

The positive value of the deflection indicates that the point of application of load P moves downward (in the direction of the unit force).

Had we directed the unit force upward, we would have had $M_i = 1 \times z$ and integration would have yielded the deflection with the minus sign. The negative sign would have indicated that the displacement is produced not upward but downward, as is actually the case.

Mohr's integral may be calculated by multiplying the diagrams according to Vereshchagin's rule.

Since both diagrams are straight it makes no difference from which diagram the area is taken and from which the ordinate.

The area of the loading diagram is

$$\omega_P = \frac{1}{2} Pl^2.$$

The centroid of this diagram is $^1/_3\ l$ distant from the fixed end.

Determine the ordinate through the centroid of the loading diagram to the moment diagram due to the unit force. It can readily be shown that it is equal to $^2/_3\ l$.

Therefore,

$$y = \frac{1}{EJ}\omega_P M_i^c = \frac{1}{EJ}\frac{1}{2}Pl^2\frac{2}{3}l = \frac{Pl^3}{3EJ}.$$

The same result is obtained by using the table of integrals. The result of multiplying the diagrams is positive because both diagrams are located beneath the bar. Consequently, the point of application of the load moves downward, i.e., in the assumed direction of the unit force.

To determine the angular displacement (the angle of rotation) choose the fictitious state of the beam in which a concentrated moment equal to unity acts at the end of the beam.

Construct the bending moment diagram for this case (Fig. 7.15*c*).

Determine the angular displacement by multiplying the diagrams. The area of the loading diagram is

$$\omega_P = \frac{1}{2}Pl^2.$$

The ordinates to the diagram due to the unit moment are equal to unity throughout the length. Consequently, the angle of rotation in question is

$$\theta = \frac{Pl^2}{2EJ}.$$

Since both diagrams are located below, the result of multiplying the diagrams is positive. Thus, the end section of the beam rotates clockwise (in the direction of the unit moment).

Fxample 7.7. By the use of the Mohr-Vereshchagin method determine the deflection at point *D* for the beam represented in Fig. 7.16.

Solution. Construct the separated moment diagram due to the load, i.e., construct separate diagrams due to the action of each load. For convenience in multiplying diagrams it is advisable to construct separated (elementary) diagrams in relation to the section the deflection at which

is being determined, in relation to the section D in the present case.

Fig. 7.16*a* presents the bending moment diagrams due to the reaction A (portion AD) and due to the load

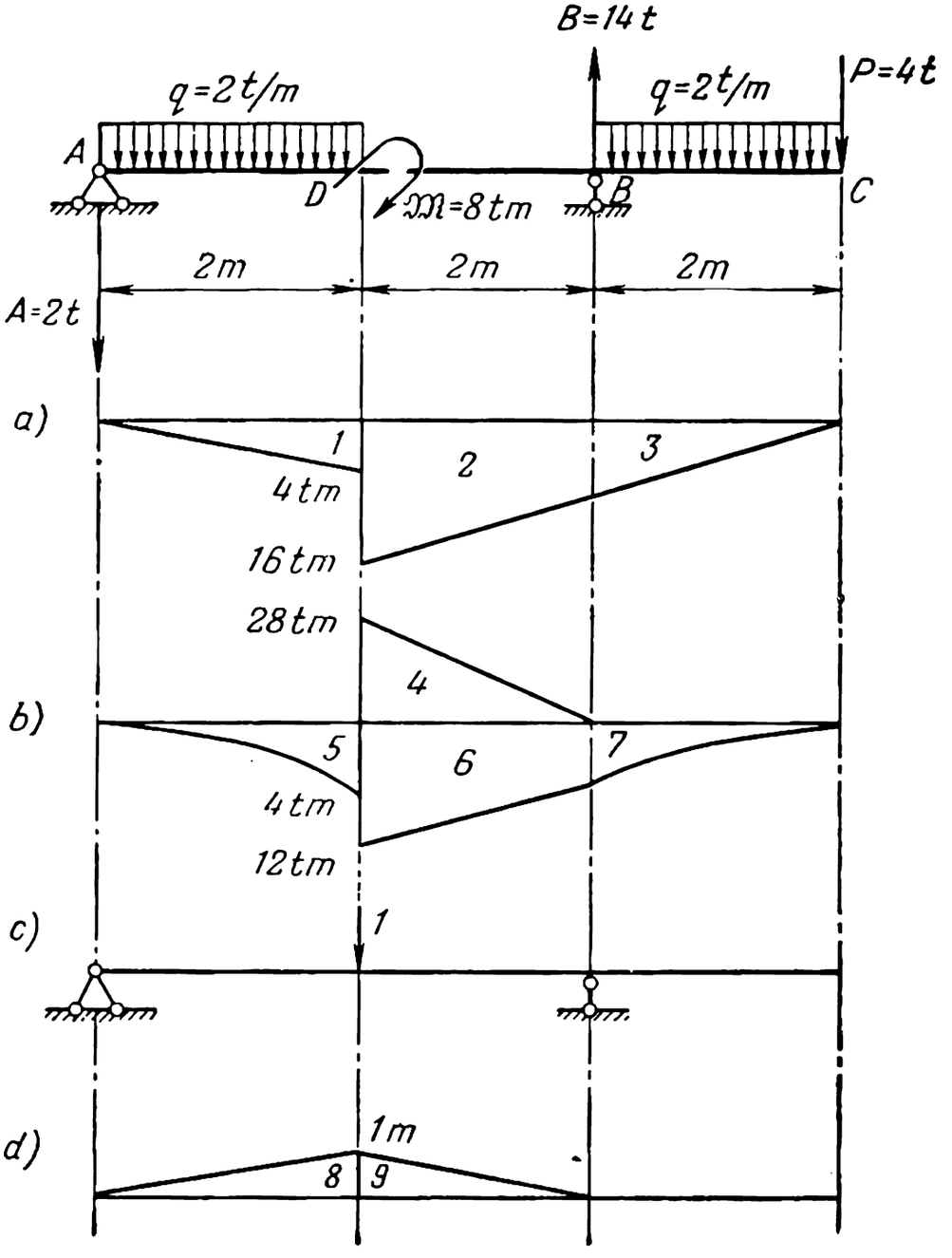


Fig. 7.16

$P=4$ tons (portion DC). The diagrams are constructed on the compressed fibre.

Fig. 7.16*b* presents the moment diagrams due to the reaction B (portion BD), due to the left-hand uniformly distributed load (portion AD) and due to the uniformly

distributed load acting in the portion *BC*. This diagram is represented in Fig. 7.16*b* in the portion *DC*, below.

Next we choose the subsidiary state of the beam, applying the unit force at the point *D* where the deflection is being determined (Fig. 7.16*c*). The moment diagram due to the unit force is represented in Fig. 7.16*d*. We now multiply diagrams *1* through *6* by diagrams *7* and *8* by using the tables for multiplying diagrams with due regard to signs. Diagrams located on the same side of the beam are multiplied with the plus sign while diagrams located on either side of the beam are multiplied with the minus sign.

Multiplying diagrams *1* and *8*, we obtain

$$\omega_1 M_8^c = -\frac{h_3 l}{3} = -\frac{4 \cdot 1 \cdot 2}{3} = -\frac{8}{3} \text{ tm}^3.$$

Multiplying diagrams *5* and *8*, we obtain

$$-\frac{f_1 l h_3}{4} = -\frac{4 \cdot 2 \cdot 1}{4} = -2 \text{ tm}^3.$$

Multiplication of diagrams *2* and *9* gives

$$-\frac{hl}{6}(2h_4 + h_5) = -\frac{1 \cdot 2(2 \cdot 16 + 8)}{3} = -\frac{40}{3} \text{ tm}^3.$$

Multiply diagrams *4* and *9*:

$$\frac{hh_3 l}{3} = \frac{1 \cdot 28 \cdot 2}{3} = \frac{56}{3} \text{ tm}^3.$$

Multiply diagrams *6* and *9*:

$$-\frac{hl}{6}(2h_4 + h_5) = -\frac{1 \cdot 2}{6}(2 \cdot 12 + 4) = -\frac{28}{3} \text{ tm}^3.$$

Summing up the results of multiplying the diagrams, we obtain

$$EJy_D = -8.67 \text{ tm}^3.$$

The negative sign indicates that the point *D* moves upward rather than downward, in the direction of the unit force.

The same result has been obtained previously by the universal formula (see Example 7.5).

Of course, in the present example it is possible to separate the diagram just in the portion *AD* as in the portion *DB* the total diagram is straight and need not be separated. In the portion *BC* the separation is not required because the diagram due to the unit force is zero in this

portion. The separation of the diagram in the portion BC is necessary for determining the deflection at the point C.

Example 7.8. Determine the vertical, horizontal and angular displacement of section A of the broken bar presented in Fig. 7.17*a*. The rigidity of the section of the vertical portion of the bar is EJ_1, that of the horizontal portion EJ_2.

Solution. Construct the bending moment diagram due to the load. It is presented in Fig. 7.17*b* (see example 6.9). To determine the vertical displacement of the section A choose the subsidiary state of the system represented in Fig. 7.17*c*. The unit vertical force acting downward is applied at point A.

The bending moment diagram for this state is given in Fig. 7.17*c*.

Determine the vertical displacement by Mohr's method, using the procedure of multiplying the diagrams. Since there is no M_1 diagram on the vertical bar in the subsidiary state, we multiply only diagrams relating to the horizontal bar. The area of the diagram is taken from the state of loading and the ordinate from the subsidiary state:

$$\Delta_{\text{vert}} = \frac{1}{EJ_2} Pl_1 l_2 \frac{1}{2} l_2.$$

Since both diagrams are located below, the result of multiplication is taken with the plus sign. Consequently, point A moves downward, i.e., in the direction of the unit vertical force.

To determine the horizontal displacement of point A choose the subsidiary state with the horizontal unit force directed to the left (Fig. 7.17*d*). The moment diagram for this case is given in the same figure.

Multiply the M_P and M_2 diagrams:

$$\Delta_{\text{hor}} = \frac{1}{EJ_1} \frac{1}{2} Pl_1 l_1 \frac{2}{3} l_1 + \frac{1}{EJ_2} Pl_1 l_2 l_1.$$

The result of multiplying the diagrams is positive because these diagrams are located on one and the same side of the bars.

To determine the angular displacement choose the subsidiary state of the system as in Fig. 7.17*e* and construct

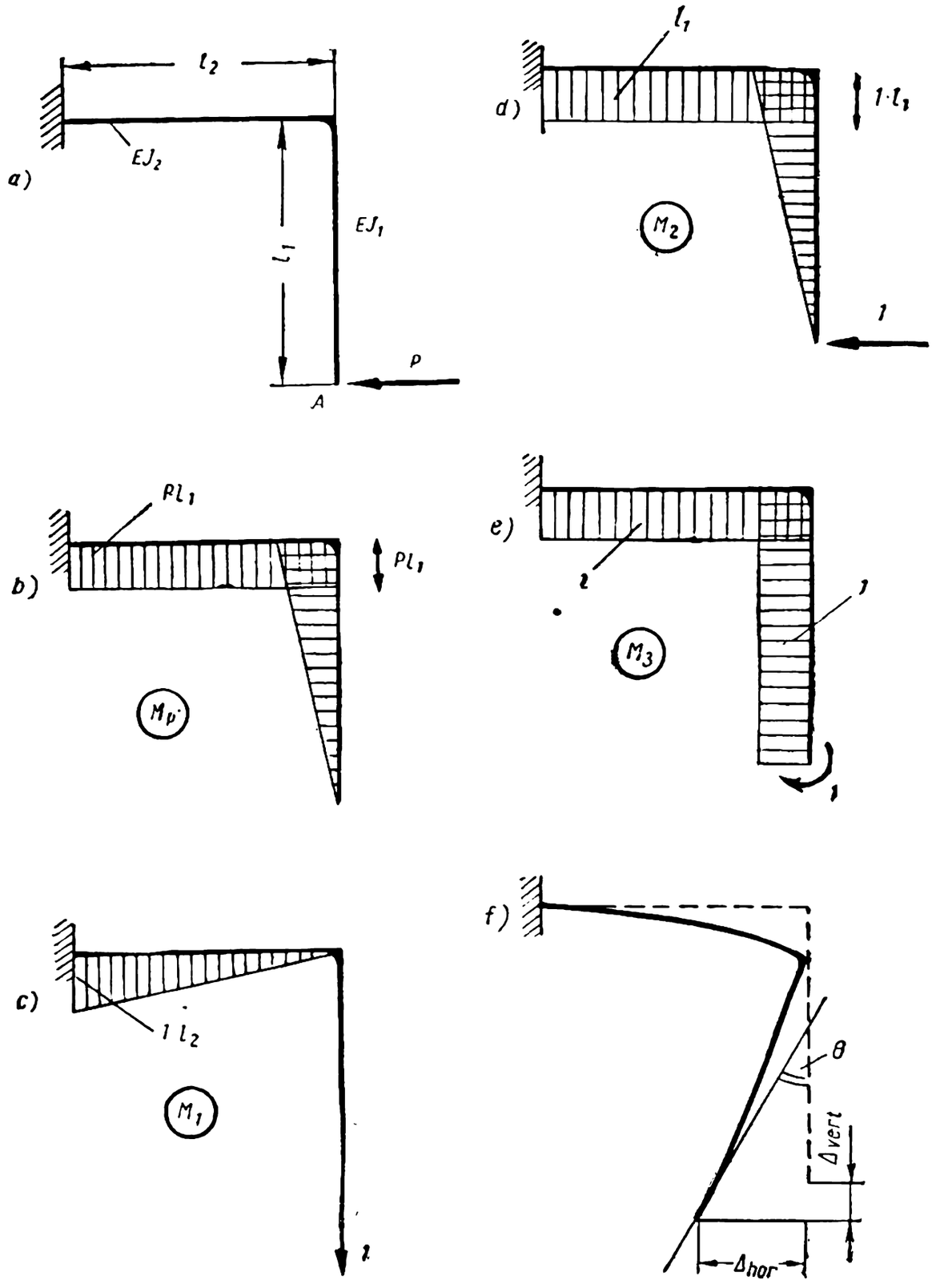

a)
l_2
EJ_2
EJ_1
l_1
P
A
d)
l_1
$1 \cdot l_1$
M_2
1
b)
Pl_1
Pl_1
M_p
e)
2
1
M_3
1
c)
$1\ l_2$
M_1
1
f)
θ
Δ_{vert}
Δ_{hor}

Fig. 7.17

the bending moment diagram for this state (in the same figure). Multiply the M_P and M_3 diagrams:

$$\theta = \frac{1}{EJ_1} \frac{1}{2} Pl_1 \times l_1 \times 1 + \frac{1}{EJ_2} Pl_1 \times l_2 \times 1.$$

The result of multiplication is positive because the diagrams are located on one side. Consequently, section A rotates clockwise. The same results would be obtained by using the tables for multiplying diagrams.

The shape of the deformed bar is shown in Fig. 7.17*f*, the displacements being greatly exaggerated.

63. Analysis of Statically Indeterminate Beams

Beams for which the equations of equilibrium alone are insufficient to determine the internal forces are called statically indeterminate.

For the analysis of such beams it is necessary to derive, besides the equations of statics, additional equations called the equations of displacements (or the equations of deformations). They are obtained from a consideration of the conditions of deformation of the beam.

Consider, for instance, the beam represented in Fig. 7.18*a*. The number of unknown reactions of the supports is four: three reactions at the built-in end and one reaction at the movable support. There are three equations of statics. Thus, there is one redundant unknown. The beam is once statically indeterminate. Redundant unknowns in problems of this type are due to the presence of constraints which are superfluous for equilibrium in a perfectly rigid body.

For the analysis of statically indeterminate beams we may apply the method already familiar to us, that was employed in solving statically indeterminate problems in tension and torsion. For that we imagine the given statically indeterminate beam to be transformed into a statically determinate one by removing the redundant constraints and replacing their action by unknown reactions.

The statically determinate system thus obtained will be called the *basic* system. So that the basic system will not differ from the given one, it is necessary that at the

points from which the constraints have been removed the displacements of the sections in the basic system in the direction of the unknown reactions applied here be equal to zero.

With these equations expressing the conditions of compatibility of deformations of the basic system with the constraints imposed on the given statically indeterminate system it will be possible to solve the stated problem.

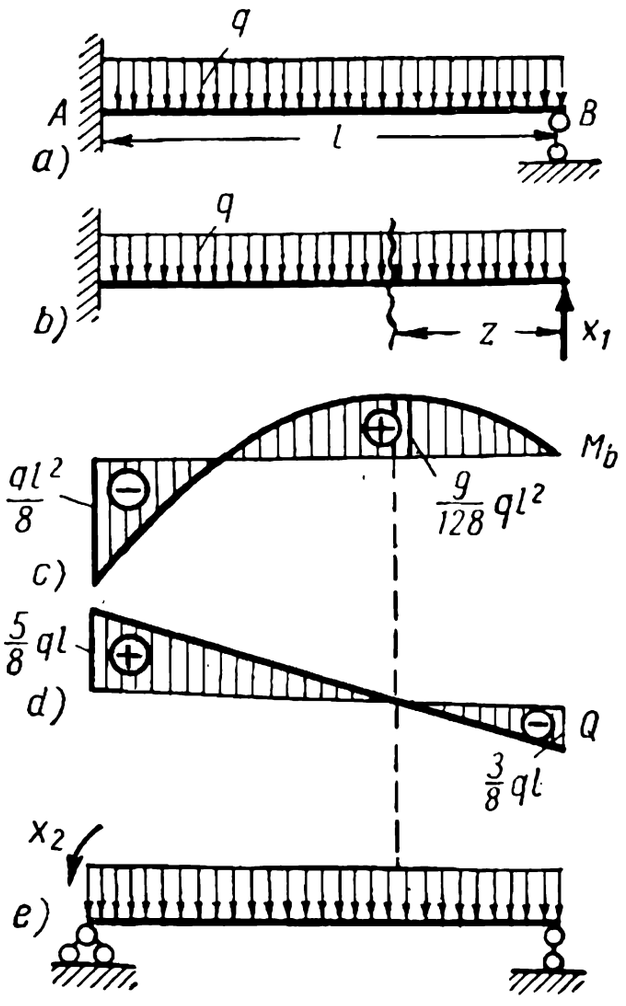


Fig. 7.18

These equations are called the *equations of compatibility of deformations.*

The basic system for one and the same statically indeterminate beam may be selected in different ways. For instance, the movable support may be removed and replaced by an unknown force X_1 (Fig. 7.18*b*). The equation of deformations in this case will express the idea that to secure equivalence of the given and basic systems the vertical displacement of the right end of the beam (point B) under the action of load q and force X_1 must be equal to zero.

To determine deflections (displacements) we may use the methods already familiar to us: the universal formula or the Mohr-Vereshchagin method.

If there is a possibility for using available formulas, advantage must, of course, be taken of such a possibility. This approach is commonly called the *method of comparing deformations.*

For instance, to derive the equations of deformations (for the basic system in Fig. 7.18*b*) use can be made of formulas (7.9) and (7.14) already encountered.

Equating the deflections of the end of the cantilever due to the uniformly distributed load and the concentrated force, we obtain

$$\frac{ql_4}{8EJ} = \frac{X_1 l^3}{3EJ}.$$

This equation yields $X_1 = \frac{3}{8} ql$. After that the M_b diagram (Fig. 7.18*c*) and the Q diagram (Fig. 7.18*d*) are constructed as in a statically determinate beam.

In the second version of the basic system (Fig. 7.18*e*) the constraint which prevents the left section from rotating is removed (the built-in end is replaced by a hinged immovable support). Its action is replaced by an unknown moment X_2. The additional equation (the equation of deformations) expresses the condition that the angle of rotation of section A under the action of load q and moment X_2 is equal to zero.

Whichever basic system is selected, it does not affect the final result of the calculation—the final M_b and Q diagrams will be identical. However, the amount of work involved in calculation differs. For beams with one redundant unknown practically the same work is involved whatever the basic system used. Where there are several redundant unknowns the basic system is chosen with a view to the solution of the system of equations of displacements involving as little computation difficulties as possible. If for a multi-span beam (such beams are often called *continuous*, Fig. 7.19) the reactions of the supports that have been removed are chosen as redundant unknowns, all of these will enter into each equation of displacements, leading to great difficulties in solving the system of equations. With a rational choice of the basic system the unknowns in the equations of displacements will be separated, i.e., a lesser number of unknowns will enter into each equation. Of course, the overall number of equations will remain unchanged (equal to the number of redundant unknowns), but instead of one system of equations with a great number of unknowns several simpler systems will have to be solved.

Thus, in the above case a big simplification will result if internal bending moments at the sections passing through the intermediate supports (the so-called support

moments) are taken as redundant unknowns, as is shown in Fig. 7.19*b*. In the basic system hinges are "cut into" the beam over each intermediate support and the violated constraints between the spans of the beam (the constraints which prevent the sections on either side of the support from rotating with respect to each other) are replaced by unknown support moments.

The equation of displacements will express the condition that the angles of rotation of the sections over the supports

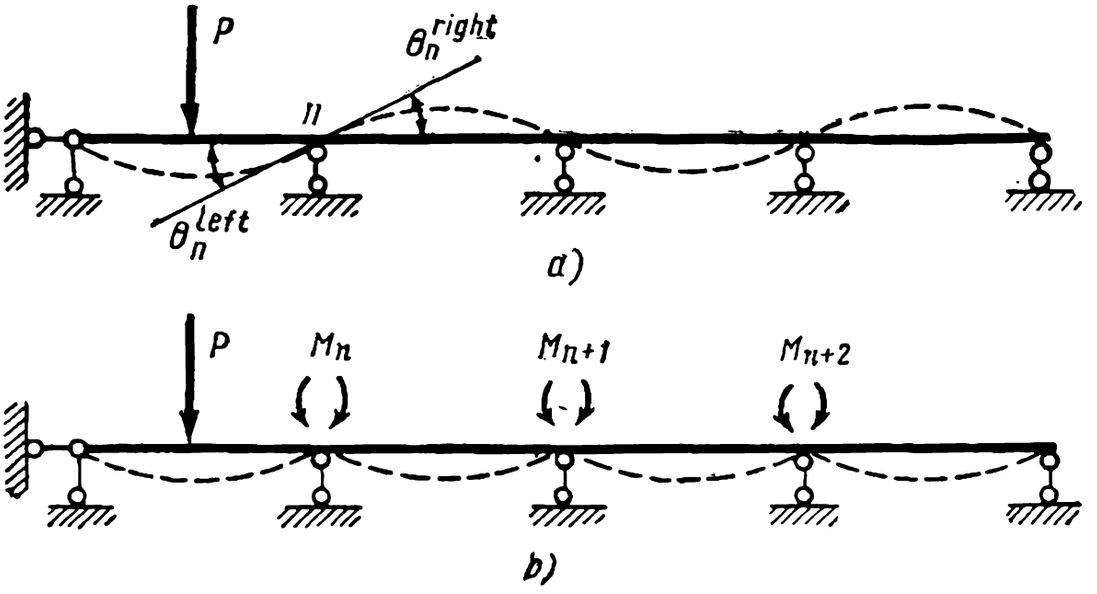


Fig. 7.19

in adjacent spans of the beam are equal to each other; for instance, for the section over the support n (see Fig. 7.19*a*):

$$\theta_n^{left} = \theta_n^{right}.$$

If these equations are expanded, it can be seen that any one of them will include not more than three support moments.

64. Examples of Calculating Statically Indeterminate Beams

Example 7.9. Plot the M_b and Q diagrams for the beam shown in Fig. 7.20*a*.

Solution. There is one redundant constraint in this construction, as is readily seen. The basic system is chosen as a cantilever beam, the right support being removed and

replaced by an unknown reaction X (Fig. 7.20*b*). The equation of deformations is: $y_B=0$.

The deflection at point B due to force X (Fig. 7.20*c*) is (see formula 7.9)

$$y_{BX}=\frac{Xl^3}{3EJ}.$$

To determine the deflection at point B due to force P (Fig. 7.20*d*) let us make use of the universal formula.

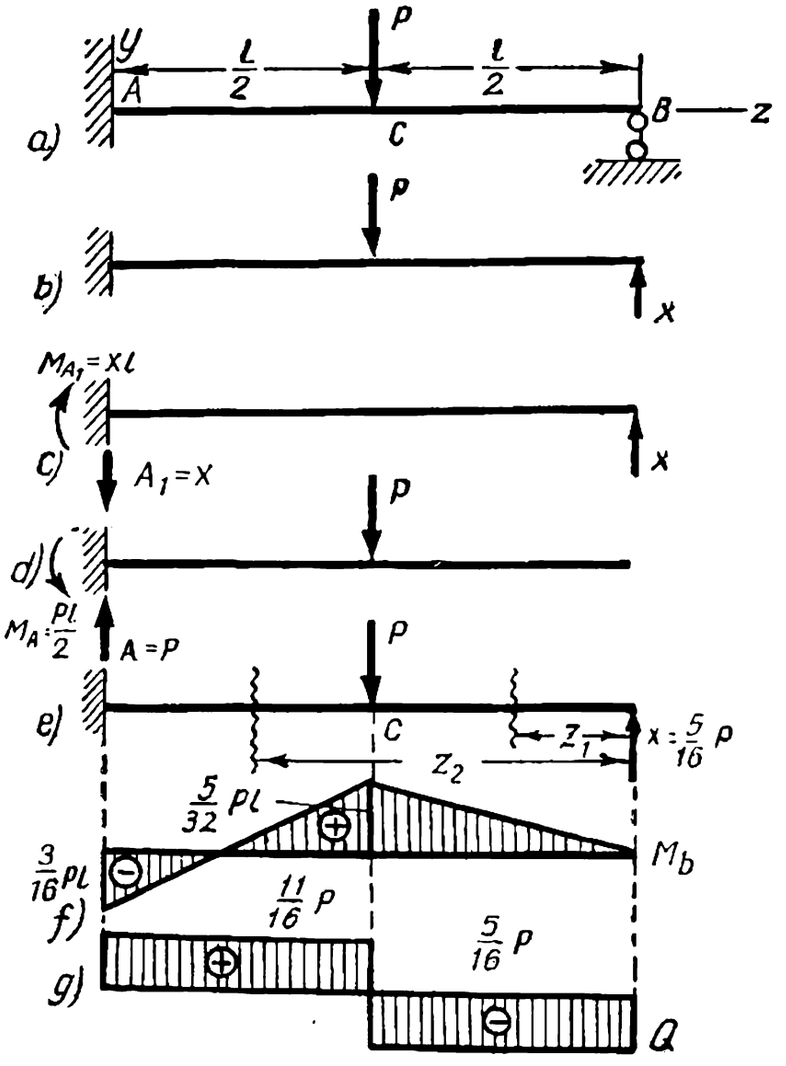


Fig. 7.20

Place the origin of co-ordinates at the left end of the beam, then $y_0=0$ and $\theta_0=0$.

Determine the reactions of the supports: $A=P$ and $M_A=P\frac{l}{2}$ (see Fig. 7.20*d*). Write the universal formula, putting $z=l$:

$$EJy_{BP}=-\frac{M_A(l-0)^2}{2}+\frac{A(l-0)^3}{6}-\frac{P\left(l-\frac{l}{2}\right)^3}{6}.$$

Substituting the value of $M_A = \frac{Pl}{2}$ and $A = P$ we obtain

$$y_{BP} = -\frac{5Pl^3}{48EJ}.$$

The equation of deformations will now be written in the form

$$\frac{Xl^3}{3EJ} - \frac{5Pl^3}{48EJ} = 0,$$

whence

$$X = \frac{5}{16}P.$$

The bending moment at any section of portion CB at a distance z_1 from the right end will be (Fig. 7.20*e*)

$$M_{z_1} = Xz_1 = \frac{5}{16}Pz_1.$$

In the portion AC

$$M_{z_2} = \frac{5}{16}Pz_2 - P\left(z_2 - \frac{l}{2}\right).$$

The M_b diagram is shown in Fig. 7.20*f*.

The shearing force in the portion CB is

$$Q_{z_1} = -X = -\frac{5}{16}P.$$

In the portion AC

$$Q_{z_2} = -\frac{5}{16}P + P = \frac{11}{16}P.$$

The Q diagram is shown in Fig. 7.20*g*.

Example 7.10. Plot the M_b and Q diagrams for the beam shown in Fig. 7.21*a*.

Solution. The problem is once statically indeterminate; take as the redundant unknown reaction X of the middle support. The equation of deformations is:

$$y_B = 0.$$

In the solution of this example, unlike the preceding one, neither the basic system loaded by the given forces alone nor the basic system loaded by unknown reaction X alone are shown in the drawing.

The deflection at point B due to load q is determined by formula (7.15)

$$y_{Bq} = -\frac{5}{384}\frac{q(2l)^4}{EJ}.$$

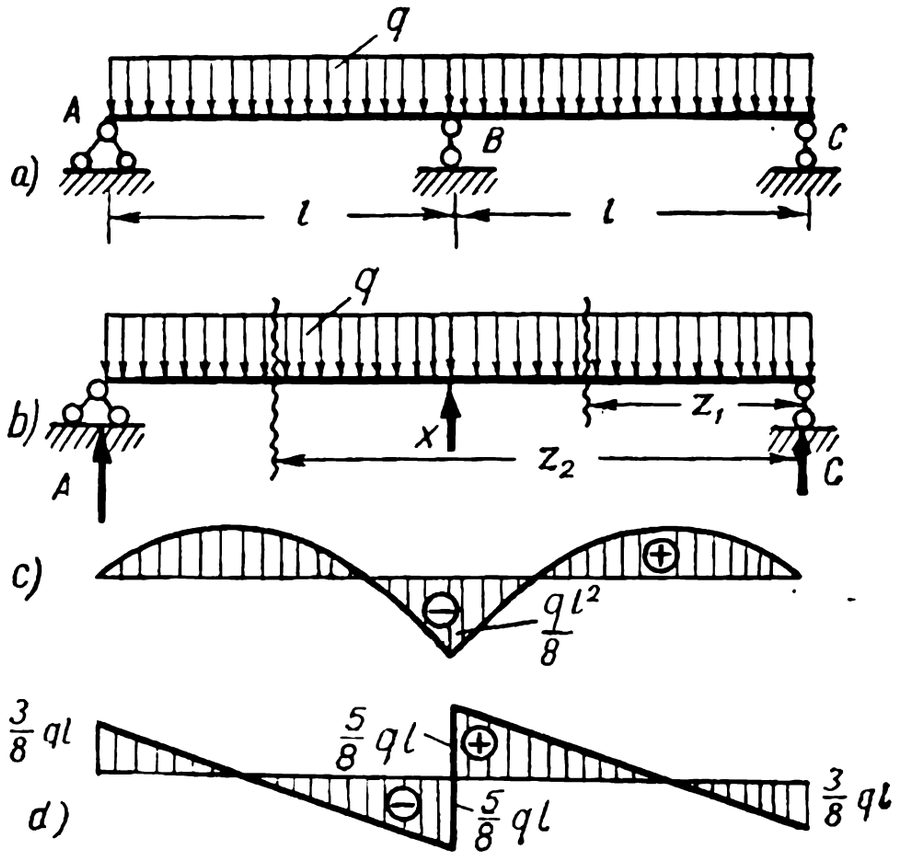


Fig. 7.21

The deflection due to force X is determined by formula (7.16)

$$y_{BX} = \frac{X(2l)^3}{48EJ}.$$

The equation of deformations becomes

$$\frac{-5q(2l)^4}{384EJ} + \frac{X(2l)^3}{48EJ} = 0,$$

whence $X = \frac{5}{4}ql$.

The reactions A and C are equal to

$$A = C = ql - \frac{5}{8}ql = \frac{3}{8}ql.$$

The bending moment at any section z_1 of the portion BC is

$$M_{z_1} = \frac{3}{8}qlz_1 - \frac{qz_1^2}{2}.$$

On the basis of this equation construct the M_b diagram (Fig. 7.21*c*). For the left half of the beam the diagram is symmetrical.

The shearing force in the portion BC is

$$Q_{z_1} = -\frac{3}{8} ql + qz_1.$$

In the portion AB

$$Q_{z_2} = -\frac{3}{8} ql - \frac{5}{4} ql + qz_2.$$

On the basis of these equations construct the Q diagram (Fig. 7.21*d*).

65. Principles of General Method of Analysis of Statically Indeterminate Systems (Principles of Force Method)

In the preceding sections statically indeterminate beams have been considered.

In engineering practice more complicated statically indeterminate systems are encountered, i.e., *systems in which the internal forces cannot be determined from the equations of equilibrium alone.*

Statically indeterminate systems composed of rigidly connected bars are called *frames.* The methods of analysis of statically indeterminate bar systems are treated in detail in Structural Mechanics.

These methods are widely different. We shall present here only the principles of one of the more commonly used methods, *the force method*, and illustrate it by examples of the simplest statically indeterminate systems.

The order of calculating statically indeterminate systems by the force method is as follows.

1. First find the degree of statical indeterminacy of a system by determining the number of redundant constraints.

2. Next select *the basic system*, which is obtained from the given system after removing the redundant constraints. The constraints are replaced by redundant unknown forces.

3. Derive the equations of deformations (of displacements, to be more exact) which express the conditions of compatibility of deformations of the basic system with

the given statically indeterminate system. If the displacements in the direction of the removed constraints in the basic system must be zero, the equations of deformations express the condition that these displacements are zero.

4. Solve the derived equations.

5. When the redundant unknowns have been found determine the internal forces in the elements of the statically indeterminate system (bending moments, shearing forces, etc.). This can be done without difficulty by means of the method of sections.

The equations of displacements in the analysis of statically indeterminate systems by the force method are written in a certain (*canonical*) form.

If a given statically indeterminate system has n redundant unknowns, the system of n canonical equations for determining these unknowns is written in the following general form:

$$\begin{aligned}
&\delta_{11}X_1+\delta_{12}X_2+\ \ldots\ +\delta_{1n}X_n+\Delta_{1p}=0,\\
&\delta_{21}X_1+\delta_{22}X_2+\ \ldots\ +\delta_{2n}X_n+\Delta_{2p}=0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\delta_{n1}X_1+\delta_{n2}X_2+\ \ldots+\delta_{nn}X_n+\Delta_{np}=0.
\end{aligned}$$

In the first equation δ_{11} is the displacement of the point of application of the first redundant unknown in its own direction caused by the unit value of this unknown. The product $\delta_{11}X_1$ represents the displacement of the same point in the same direction caused by force X_1.

The second term $\delta_{12}X_2$ represents the displacement of the same point in the same direction caused by force X_2, etc.

The term Δ_{1p} represents the displacement of the same point and in the same direction caused by the load.

The whole left-hand member of the first equation represents the total displacement of the point of application of force X_1 in the direction of this force caused by all forces.

This displacement is set equal to zero since there is none in the given system.

The second equation expresses the condition that the total displacement of the point of application of force X_2 in the direction of this force due to all acting forces is equal to zero.

The meaning of the remaining equations is clear.

Difficulties in the analysis of statically indeterminate systems lie in the solution of these equations rather than in their derivation.

When the number of equations is not great they can be readily solved by the method of successive elimination of unknowns. When a great number of equations is involved special procedures are employed to simplify their solution (the Hauss method, the method of successive approximations, etc.); these questions are treated in more detail in Structural Mechanics.

The technique for calculating the simplest statically indeterminate systems will be illustrated by concrete examples.

Example 7.11. Design the frame represented in Fig. 7.22*a*. The sections of bars have the same rigidity.

Solution. This frame is once statically indeterminate. In fact, there are in all four unknown reactions of the supports (three at the fixed end and one at the right-hand support). There are, however, only three equations of equilibrium.

The basic system is taken to be the system shown in Fig. 7.22*b* which is obtained from the given one by removing the right-hand support. The redundant unknown which compensates for the constraint is denoted by X_1. Derive the equation of displacements by equating to zero the displacement in the direction of X_1.

In this case the canonical equation will be

$$\delta_{11}X_1 + \Delta_{1q} = 0.$$

Here δ_{11} is the displacement in the direction of the first unknown due to the unit value of the same unknown; Δ_{1q} is the displacement in the direction of the first unknown due to the load.

To determine the displacements use is made of the Mohr-Vereshchagin method (multiplying the diagrams).

Fig. 7.22*c* gives the bending moment diagram due to the load.

Fig. 7.22*d* shows the diagram of the moments $\overline{M}_1$ due to the unit force $X_1 = 1$.

To determine the displacement δ_{11} multiply the M_1 diagram by itself. We obtain

$$\delta_{11} = \frac{1}{EJ}\,\frac{1}{2}\,l \times l \times \frac{2}{3}\,l = \frac{l^3}{3EJ}.$$

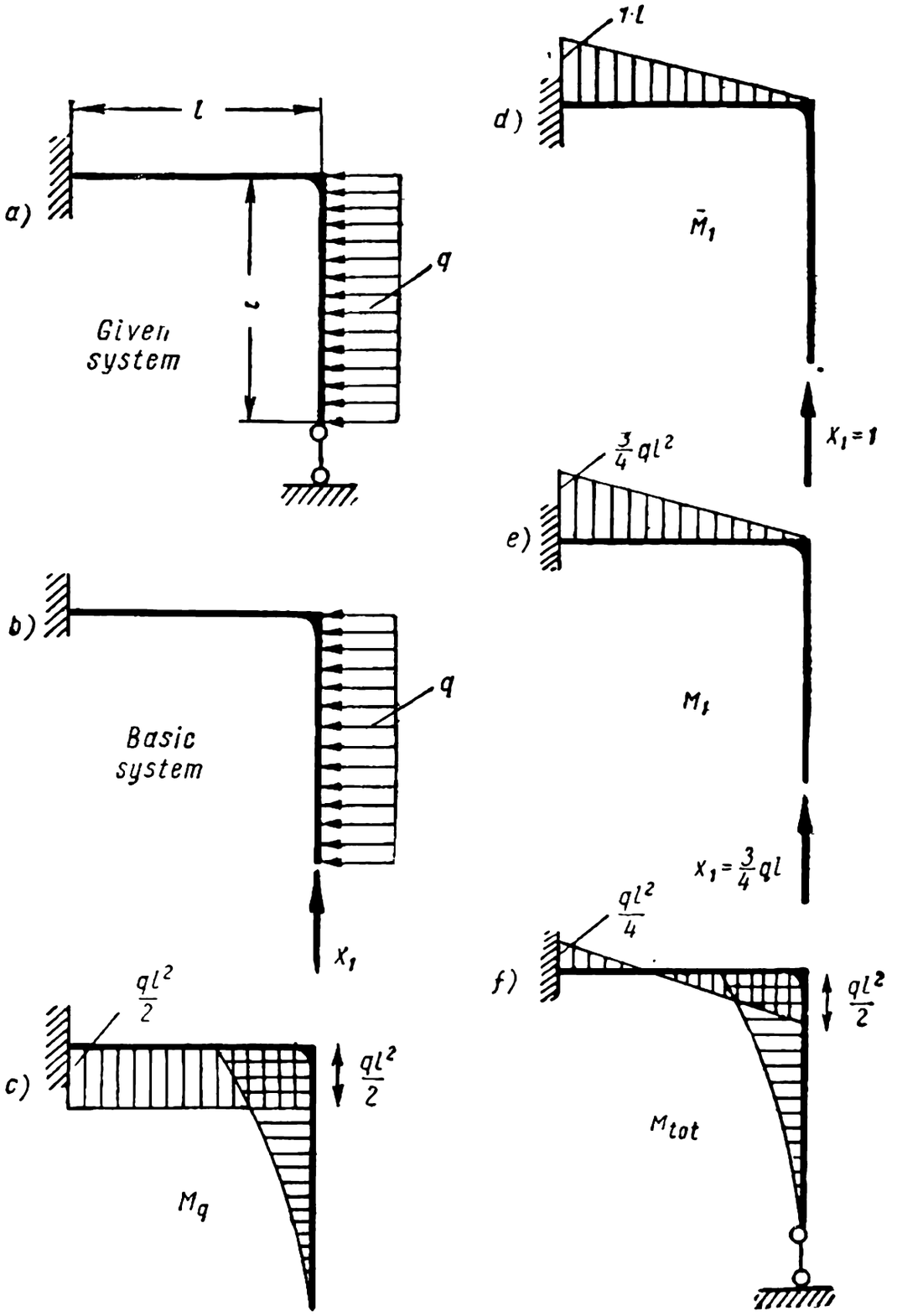

a)
l
q
Given system
b)
Basic system
q
X_1
c)
$\frac{ql^2}{2}$
$\frac{ql^2}{2}$
M_q
d)
1·l
$\bar{M}_1$
$X_1 = 1$
e)
$\frac{3}{4}ql^2$
M_1
$X_1 = \frac{3}{4}ql$
f)
$\frac{ql^2}{4}$
$\frac{ql^2}{2}$
M_{tot}

Fig. 7.22

To determine the displacement Δ_{1q} we multiply the M_q and $\overline{M}_1$ diagrams. Note that the diagrams lie on different sides of the bar (horizontal):

$$\Delta_{1q} = \frac{1}{EJ} \frac{ql^2}{2} \times l \times \frac{l}{2} = -\frac{ql^4}{4EJ}.$$

Therefore

$$\frac{l^3}{3EJ} X_1 - \frac{ql^4}{4EJ} = 0,$$

whence

$$X_1 = \frac{3}{4} ql.$$

The resulting positive value of X_1 indicates that the assumed direction for X_1 corresponds to the actual one.

The problem is solved.

It is now possible to draw the bending moment diagram for the given system. This can be done most easily as follows. It is necessary to enlarge the $\overline{M}_1$ diagram by a factor of $\frac{3}{4} ql$ (we obtain the M_1 diagram) and to add it to the M_q diagram.

The M_1 diagram is shown in Fig. 7.22*e*. By summing it up with the M_q diagram we obtain the final (total) M_{tot} diagram given in Fig. 7.22*f*.

Example 7.12. Design the frame represented in Fig. 7.23*a*.

Solution. This frame is also once statically indeterminate. The basic system is given in Fig. 7.23*b*. Fig. 7.23*c* shows the moment diagram due to the load and Fig. 7.23*d* shows the diagram due to the unit value of the redundant unknown.

The canonical equation of displacements is

$$\delta_{11} X_1 + \Delta_{1q} = 0.$$

Calculate δ_{11} by multiplying the $\overline{M}_1$ diagram by itself

$$\delta_{11} = \frac{1}{EJ} \times \frac{1}{2} l \times l \times \frac{2}{3} l + \frac{1}{EJ} l \times l \times l = \frac{4l^3}{3EJ}.$$

Calculate Δ_{1q} by multiplying the M_q and $\overline{M}_1$ diagrams. In multiplying the square parabola by the triangle (on the vertical bar) make use of the data from the table for the multiplication of diagrams:

$$\Delta_{1q} = -\frac{1}{EJ} \frac{ql^2}{2} \times \frac{l \times l}{4} - \frac{1}{EJ} \frac{ql^2}{2} \times l \times l = -\frac{5}{8} \frac{ql^4}{EJ}.$$

Therefore

$$X_1 = -\frac{\Delta_{1q}}{\delta_{11}} = \frac{15}{32}\,ql.$$

Enlarge the ordinates to the $\bar{M}_1$ diagram by a factor of $\frac{15}{32}\,ql$ (Fig. 7.23*e*) and add them to the ordinates to the M_q diagram.

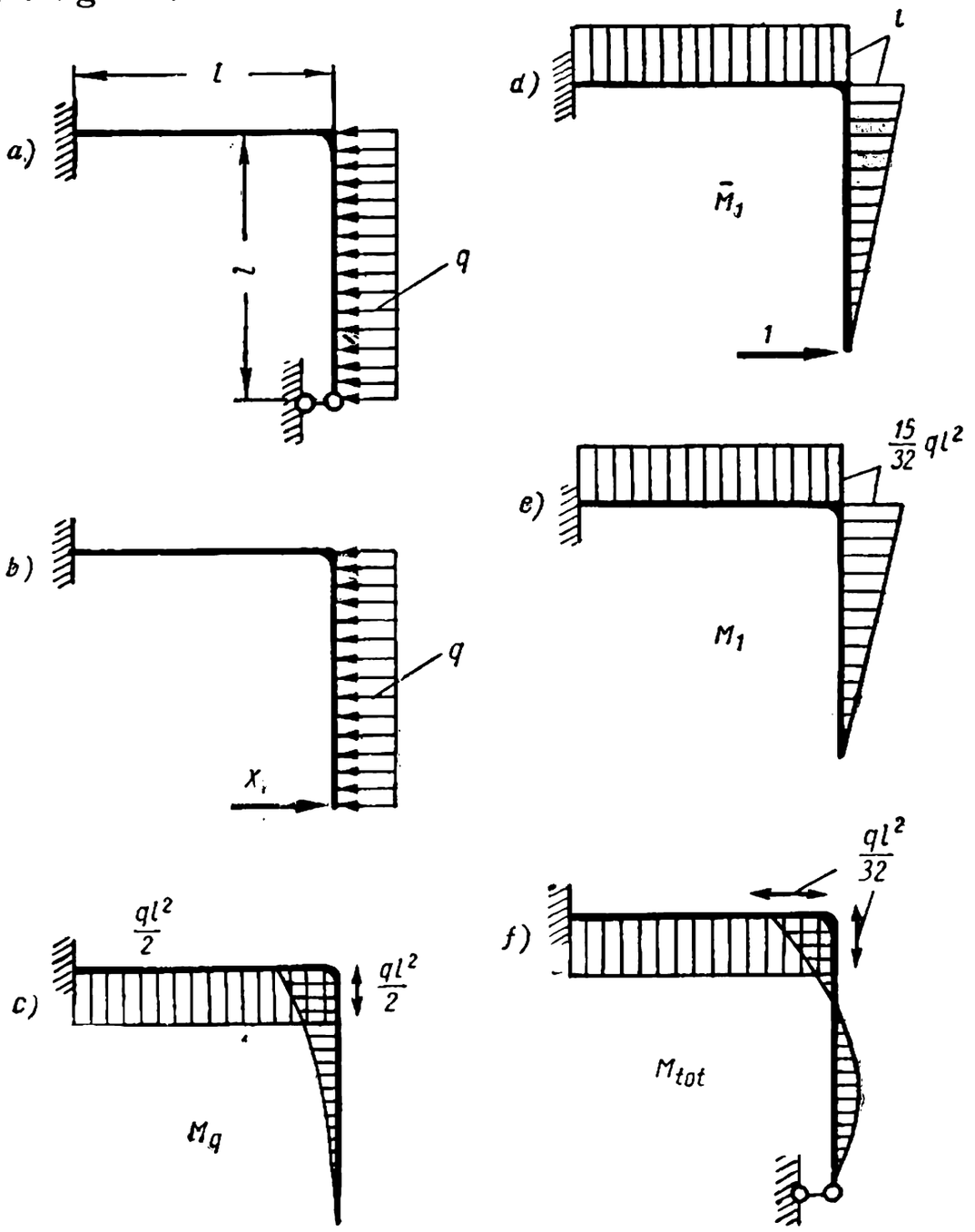


Fig. 7.23

The total moment diagram is given in Fig. 7.23*f*.

Example 7.13. Design the frame represented in Fig. 7.24*a*.

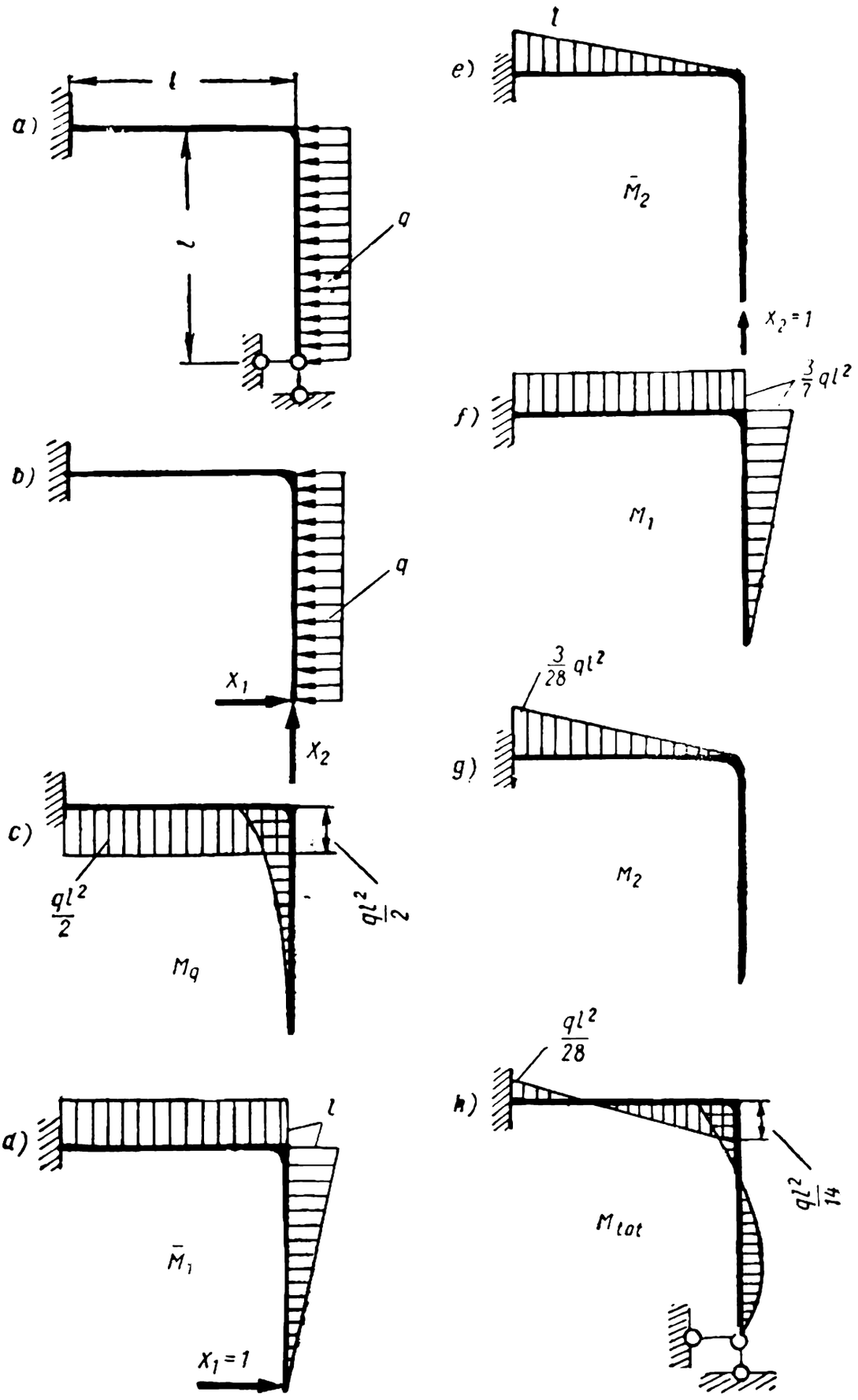

a)
l
l
q
b)
q
X1
X2
c)
ql²/2
ql²/2
Mq
d)
l
M̄1
X1 = 1
e)
l
M̄2
X2 = 1
f)
3/7 ql²
M1
g)
3/28 ql²
M2
h)
ql²/28
ql²/14
Mtot

Fig. 7.24

Solution. This frame is twice statically indeterminate. As a basic system we choose the broken bar shown in Fig. 7.24*b*.

The canonical equations for a system with two unknowns are

$$\delta_{11}X_1+\delta_{12}X_2+\Delta_{1q}=0,$$
$$\delta_{21}X_1+\delta_{22}X_2+\Delta_{2q}=0.$$

The first equation expresses the condition that the displacement in the direction of the first unknown due to all acting forces is equal to zero.

The second equation expresses the condition that the displacement in the direction of the second unknown due to all acting forces is equal to zero. The coefficients in the equations are determined by multiplying the diagrams.

The displacements δ_{11} and δ_{22} are called principal, δ_{12} and δ_{21} subsidiary. On the basis of Maxwell's reciprocal theorem on displacements we have: $\delta_{12}=\delta_{21}$. The diagrams due to the load and those due to the unit values of the redundant unknowns are shown in Fig. 7.24 *c, d, e.*

Multiplying the $\overline{M}_1$ diagram by itself we obtain

$$\delta_{11}=\frac{4l^3}{3EJ}\ \text{(see the preceding example)},$$

$$\delta_{22}=\frac{l^3}{3EJ}\ \text{(ditto)},$$

$$\delta_{12}=\delta_{21}=\frac{1}{EJ}\times\frac{1}{2}\,l\times l\times l=\frac{l^3}{3EJ},$$

$$\Delta_{1q}=-\frac{5}{8}\times\frac{ql^4}{EJ},$$

$$\Delta_{2q}=-\frac{ql^4}{4EJ}.$$

Substituting the results in canonical equations and dividing by a common factor $\frac{l^3}{EJ}$ we obtain

$$\frac{4}{3}X_1+\frac{1}{2}X_2-\frac{5}{8}ql=0,$$

$$\frac{1}{2}X_1+\frac{1}{3}X_2-\frac{1}{4}ql=0.$$

Solving this system of equations we have

$$X_1 = \frac{3}{7} ql,$$

$$X_2 = \frac{3}{28} ql.$$

Enlarging the unit $\overline{M}_1$ and $\overline{M}_2$ diagrams by a factor of X_1 and X_2 respectively we obtain the M_1 and M_2 diagrams (Figs. 7.24*f* and *g*).

By summing up the M_1, M_2 and M_q diagrams we obtain the overall bending moment diagram represented in Fig. 7.24*h*.

66. Rational Arrangement of Supports of Beams

The proper arrangement of supports of beams (if there is no objection because of service or other reasons) is of prime importance from the viewpoint of economy of material. This relates both to statically determinate and to statically indeterminate beams. *

In a simple beam supported at the ends and subjected to a uniformly distributed load the maximum bending moment (Fig. 7.25*a*) occurs, as is known, at the mid-span $M_{max} = \frac{ql^2}{8}$. It is easy to check that in a beam of the same length but with overhangs (Fig. 7.25*b*) the bending moment will be reduced.

The lowest value of the maximum bending moment is obtained in a case where the moment at the section over the support is equal to the maximum moment in the span. This condition is satisfied when the length of each overhang is $0.207l$. The resulting maximum bending moment is $M_{max} = \frac{ql^2}{46.6}$.

Thus, by a proper arrangement of the supports the bending moment can in this case be decreased to about one-sixth.

The reader is recommended to determine the most advantageous length of a beam with a single overhang, subjected to a uniformly distributed load.

---

* For further details on this subject see the work of P. A. Stepin and G. A. Snesarev *Economy of Materials in Machine Design*, Moscow, Mashgiz, 1960.

In a two-span statically indeterminate beam there are three different bending moments to consider: M_A, ($M_C = M_A$), M_B and the moment in the span M_s (Fig. 7.26). In order that the beam may have the least cross section constant over the entire length two of the above three maximum moments must be equal.

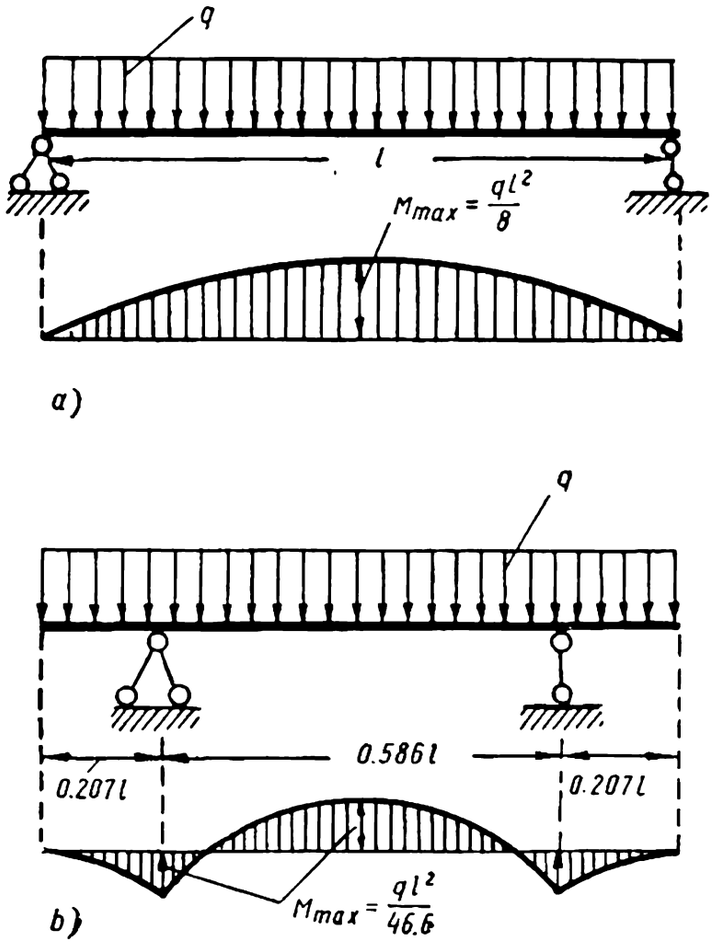


Fig. 7.25

The student is recommended to prove that the lowest value of the maximum bending moments will be obtained when the length of the overhangs is $0.408l$; in this case $M_A = M_B = \frac{ql^2}{12}$, $M_s = \frac{ql^2}{24}$.

For a two-span beam of the same length ($0.408l \times 2 + 2l$) without overhangs the maximum bending moment occurs at the section over the middle support (see Fig. 7.26 and 7.21) and is given by

$$M = \frac{q(1.408l)^2}{8} = \frac{ql^2}{4}.$$

Thus, the introduction of two overhangs in this case permits the design moment to be reduced to one-third.

Fig. 7.27*a* shows the spacing advisable for a three-span beam without overhangs, and Fig. 7.27*b* for a five-span beam.

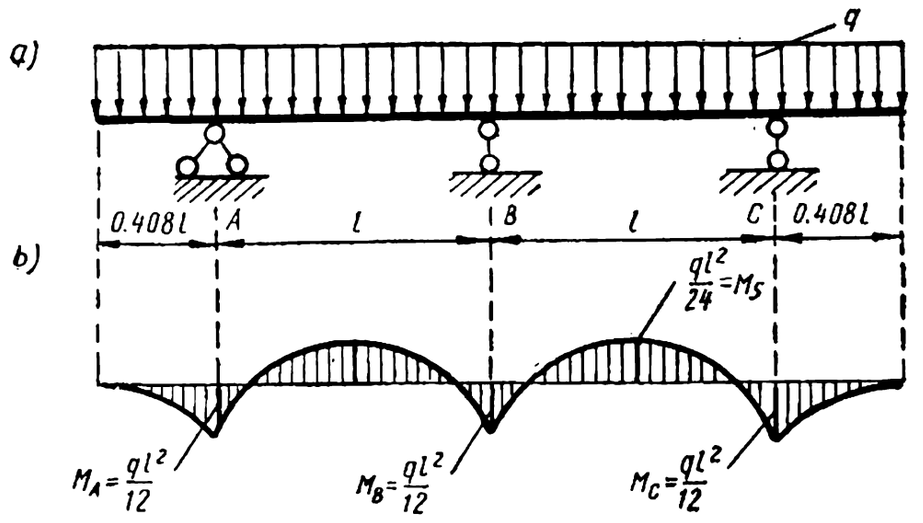


Fig. 7.26

With the appropriate analysis it is also possible to establish the optimum arrangement of supports under other types of loading. Let, for instance, any portion of a beam

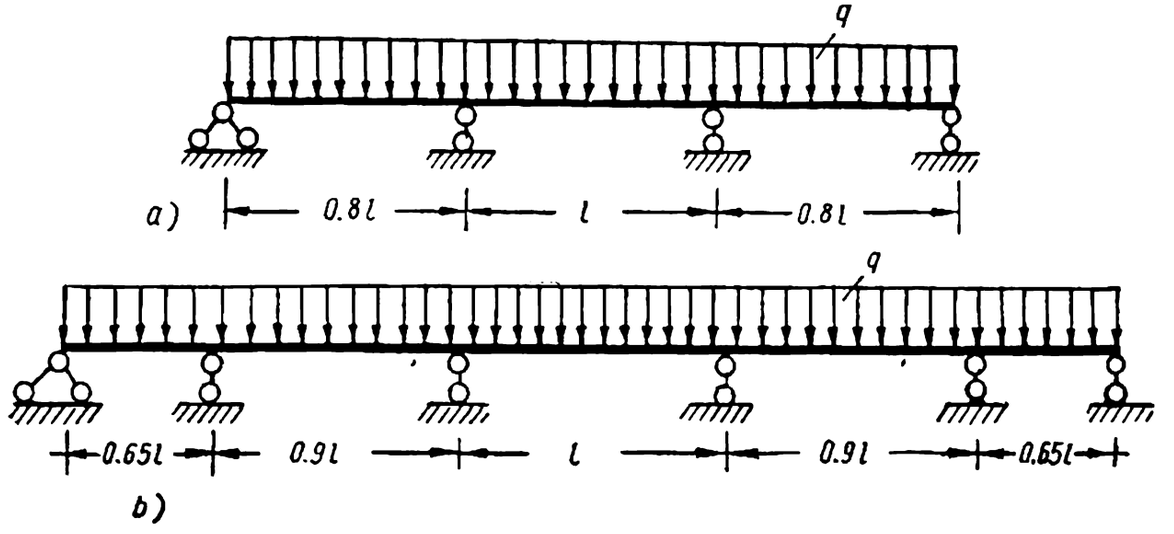


Fig. 7.27

be subjected to a uniform load of q intensity. When the load q_1 is applied to the overhangs (Fig. 7.28*a*)

$$M'_{\max}=\frac{q_1a^2}{2}.$$

When the load q_2 is applied to the portion between the supports (Fig. 7.28*b*)

$$M''_{\max}=\frac{q_2l^2}{8}.$$

By equating the maximum values of the moments we obtain the optimum value of the length of the overhang

$$a = \frac{l}{2}\sqrt{\frac{q_2}{q_1}}.$$

It should be noted that analysis based on bending moments, i.e., strength analysis, is not always decisive.

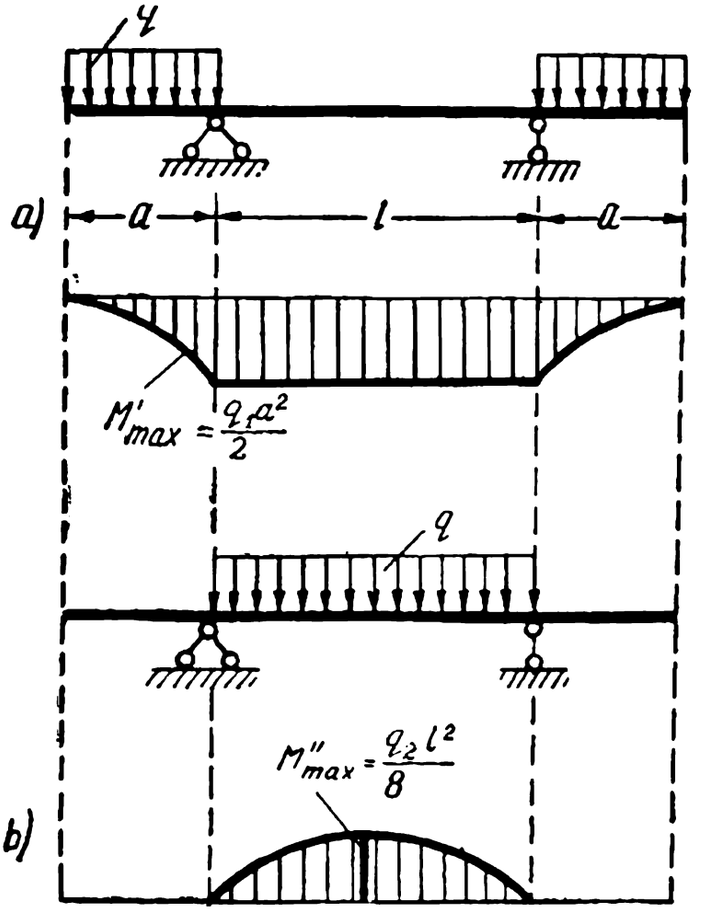


Fig. 7.28

In some cases the arrangement of supports should be governed by the condition of rigidity in order to obtain the smallest deflections. For instance, for a simple beam subjected to a uniformly distributed load (see Fig. 7.25*a*) the maximum deflection at the mid-span is

$$y = -\frac{5}{384}\frac{ql^4}{EJ} = -\frac{0.013ql^4}{EJ}.$$

With two overhangs the deflection at the mid-span of the beam will be decreased (Fig. 7.25*b*). The optimum length of the overhang will be obtained from the condition that the deflection at the end of the overhang should be equal to the deflection of the beam at the mid-span. By using

the universal formula and omitting calculations which the student is recommended to carry out we obtain the optimum length of the overhang from the condition of rigidity:

$$a = 0.223l.$$

For this length of overhang the maximum deflection of the beam will be $0.00095 \frac{ql^4}{EJ}$, which is less than the deflection of a beam without overhangs by a factor of 13.7. It is not surprising as the length of a beam enters to the fourth power in the deflection formula. So when reducing the length of a beam by a factor of 2, for example, the deflection of the beam will be decreased by a factor of 16.

67. Rational Shapes of Cross Sections of Beams

The most efficient cross sections of beams, from the viewpoint of economy of material, are those in which most of the material is put in the top and bottom parts of the section where the stresses are highest and the material is therefore most fully utilised (Fig. 7.29*a*).

For a quantitative estimation of various shapes of cross sections (based on economy of material) use can be made of the following dimensionless quantity

$$w_x = \frac{w_x}{\sqrt{F^3}}$$

called the axial unit section modulus. The magnitude of w_x depends only on the shape of the cross section.

Table 7.3 gives values of w_x for some of the most widely used sections.

Table 7.3

Values of Axial Unit Section Moduli

Type of section	w_x
Circle	0.14
Square	0.167
Ring	0.73-0.84
Channel	0.57-1.35
I-section ordinary	1.02-1.51

As is seen, the least economical are circular, square and other sections of a similar nature, in which much of the material is concentrated near the neutral axis where the material is understressed (Fig. 7.29*b*).

The most economical are I-sections, channels * and box sections (Fig. 7.29*a*).

The question of the rational depth for I-beams is of great practical importance since this section is the basic fabricated beam section.

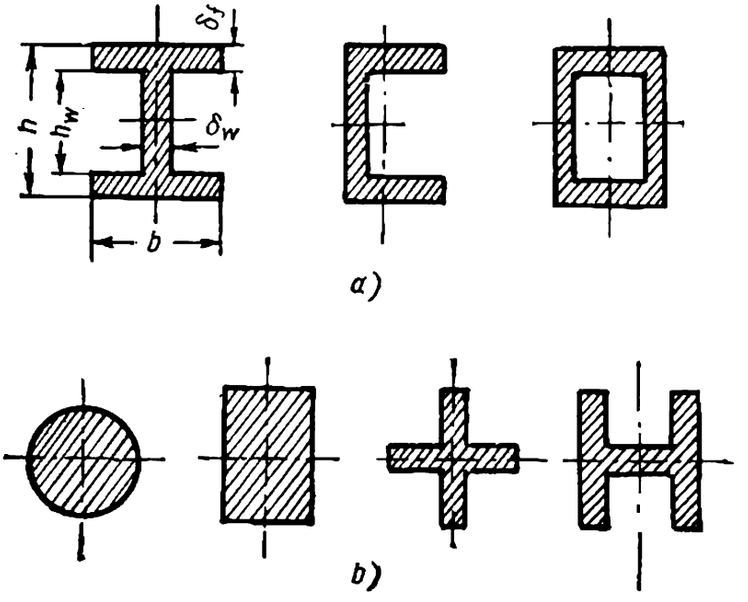


Fig. 7.29

In solving this problem the following simplifications are possible.

1. The flange thickness δ_f can be neglected as compared with the web height h_w and therefore the latter can be identified with the depth of the whole beam h (see Fig. 7.29*a*).

2. In computing the moment of inertia of the section with respect to the neutral axis the moment of inertia of the flange with respect to its own centroidal axis can be neglected.

Under these two assumptions the moment of inertia of the section is

$$J_x = 2F_f\left(\frac{h}{2}\right)^2 + \frac{\delta_w h^3}{12} = \frac{F_f h^2}{2} + \frac{\delta_w h^3}{12},$$

where F_f is the cross-sectional area of the flange.

* Provided the possibility of twisting is eliminated.

The section modulus is

$$W_x = \frac{2J_x}{h} = F_f h + \frac{\delta_w h^2}{6},$$

whence

$$F_f = \frac{W_x}{h} - \frac{\delta_w h}{6}.$$

Therefore the total cross-sectional area is

$$F = 2F_f + F_w = \frac{2W_x}{h} - \frac{2\delta_w h}{6} + \delta_w h = \frac{2W_x}{h} + \frac{2}{3}\delta_w h.$$

Now we can find the value of h which makes the cross-sectional area a minimum. For that it is necessary to establish the relation between the web thickness δ_w and the height h.

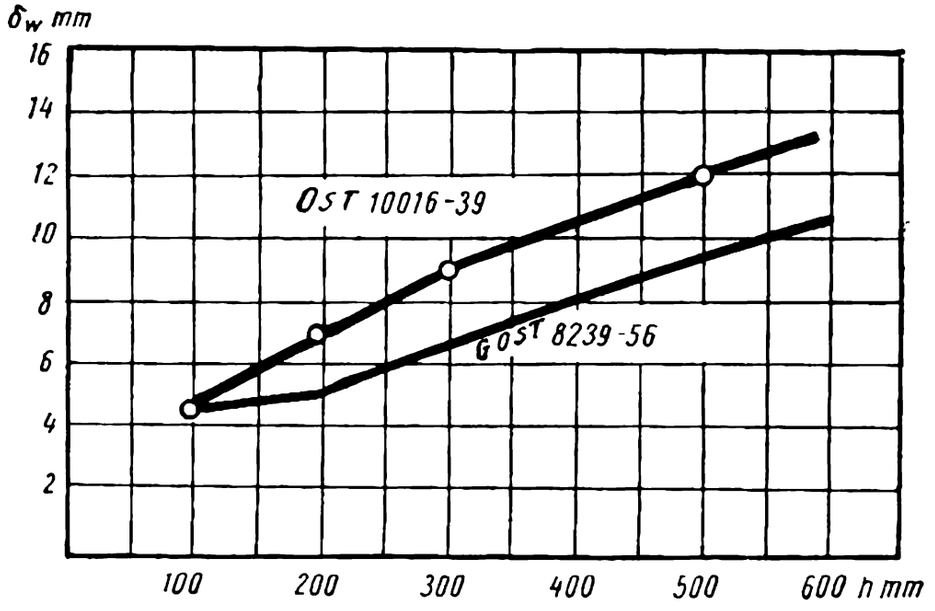


Fig. 7.30

Fig. 7.30 shows the δ_w-h relation for I-beams. It is seen from this graph that this relation can be approximated by an equation of a straight line not passing through the origin:

$$\delta_w = \alpha + \beta h. \qquad (7.34)$$

The factors α and β for beams according to GOST 8239-56 are: $\alpha = 2$ mm, $\beta = 0.0129$; according to OST 10016-39 $\alpha = 3.66$ mm, $\beta = 0.0167$. By use of relation (7.34) we obtain for the cross-sectional area

$$F = \frac{2W_x}{h} + \frac{2}{3}\alpha h + \frac{2}{3}\beta h^2. \qquad (7.35)$$

The optimum height of a beam will be found by setting the derivative $\frac{dF}{dh}$ equal to zero:

$$W_x - \frac{\alpha}{3} h_{opt}^2 - \frac{2}{3} \beta h_{opt}^3 = 0. \qquad (7.36)$$

By solving this cubic equation we determine h_{opt}. The equation can be solved by use of tables, the method of trial and error and, finally, the graphical method. For that W_x is calculated according to formula (7.36) for

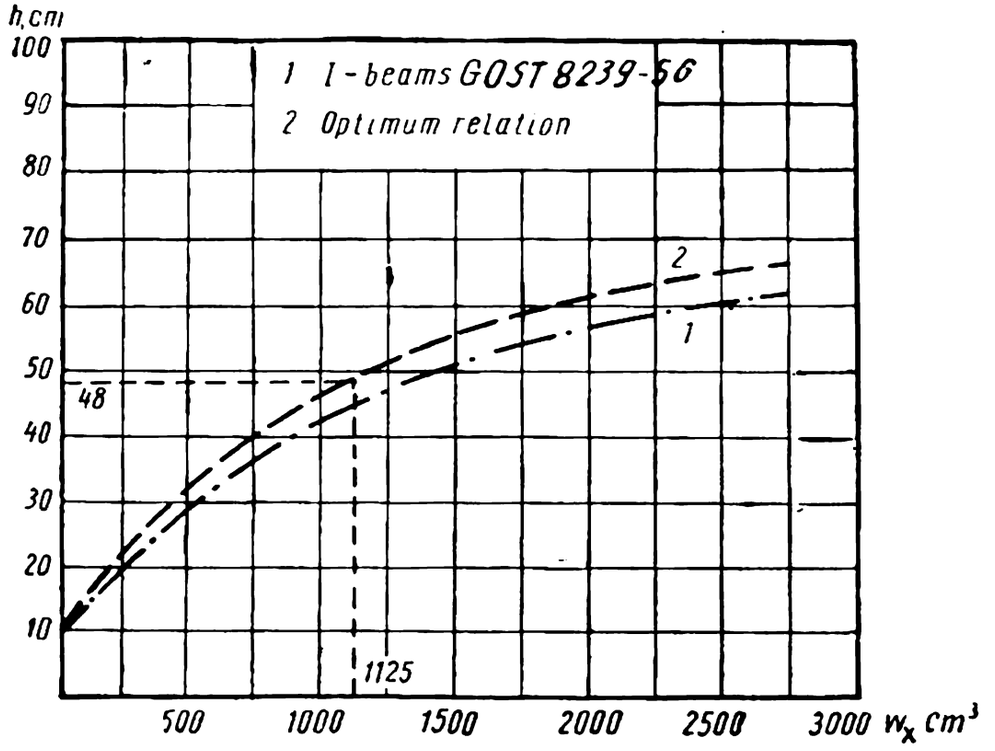


Fig. 7.31

several prescribed values of h_{opt} and the relation between these quantities is plotted. With such a graph it is easy to solve the inverse problem: determine h_{opt} for a given W_x.

In Fig. 7.31 curve *2* gives the optimum relation between W_x and h according to equation (7.36) and curve *1* gives the relation between W_x and h for I-beams according to GOST 8239-56.

It is seen from a comparison of these curves that the height of rolled beams is on the conservative side of the optimum height, which leads to over-expenditure of metal (10-20%).

At $\alpha=0$ formula (7.36) yields the approximate formula

$$h_{\text{opt}} = \sqrt[3]{\frac{3W_x}{2\beta}}, \qquad (7.37)$$

which is used, for example, in assigning the height of welded beams.

If the material of a beam has different resistance in tension and compression (for instance, cast iron), the section which is symmetrical with respect to the neutral axis becomes non-economical.

Compare, for instance, two sections of a cast-iron beam: I-section and T-section (Fig. 7.32).

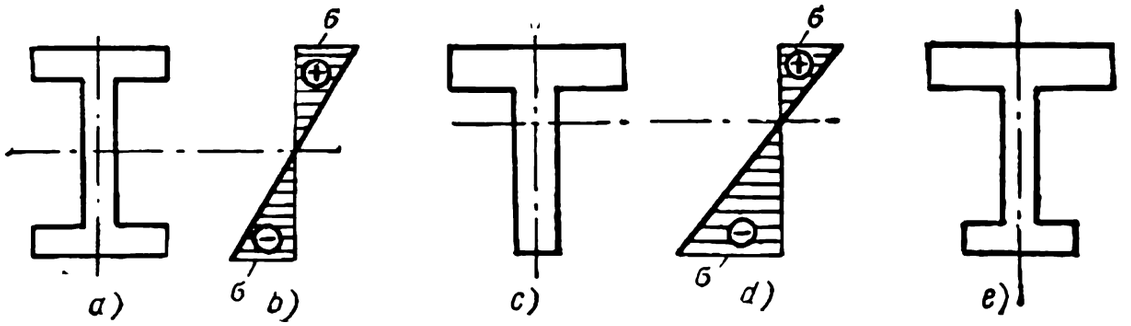


Fig. 7.32

In the I-section (Fig. 7.32*a* and *b*) the maximum stresses on the extreme fibres are equal. The dimensions of the section will be determined by the allowable tensile stress which is lower than the allowable compressive stress. The stress in the compressed zone will be less than the allowable one. The material will not be used completely.

In the T-section the neutral axis will displace towards the flange, the stress diagram will be as shown in Fig. 7.32*d*.

The stresses on the extreme fibres in the flange are less than those in the web. Therefore, when such beams are employed the strength should be checked in two cross sections: according to the maximum positive bending moment and the maximum negative bending moment.

When the bending moment diagram does not change the sign it is advisable to place the flange in the stretched zone and the web in the compressed zone. In this case it is possible to choose the dimensions of the web and flanges in such a manner that the tensile stresses σ_1 will be

equal to $[\sigma_t]$ and the compressive stresses σ_2 will be equal to $[\sigma_c]$. Thus the material will be completely utilised.

A still more rational section for cast-iron beams as regards expenditure of material is that shown in Fig. 7.32*e*. The web thickness is taken minimum as given by an analysis based on shearing and principal stresses. By properly proportioning the flanges it is possible to obtain the necessary displacement of the zero line so that the stresses on the extreme fibres will be equal to the allowable tensile and compressive stresses.

In the selection of beam sections it should also be kept in mind that bending moments may vary along the length of a beam. Therefore, with a view to economising on material it is advisable to employ beams of variable cross section (Fig. 7.33).

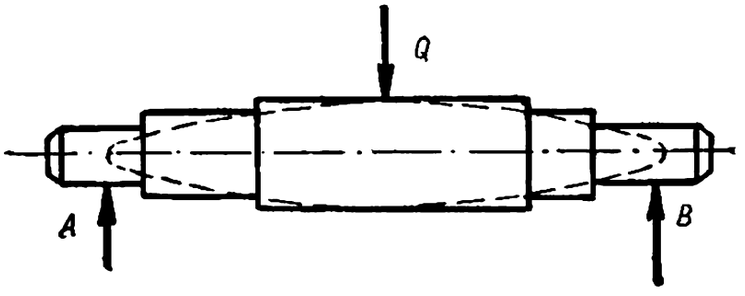


Fig. 7.33

The foregoing method of analysing efficiency of sections based on strength can also be used for analysing efficiency of sections from the viewpoint of rigidity, i.e., from the viewpoint of obtaining the smallest cross-sectional area with a given moment of inertia.

Not being in a position to treat this important and interesting question in the concise course we suggest that the student should investigate it on his own.

In particular, the student is recommended to prove that for an I-section the optimum depth based on rigidity is given by the equation

$$\frac{2}{3}\beta h_{\text{opt}}^4 + \frac{\alpha}{3} h_{\text{opt}}^3 - 4J_x = 0.$$

The values of α and β are the same as above.

The student is also recommended to determine the optimum depth of a T-section.

Example 7.14. Choose a beam having the optimum I-section for the example 6.10 considered in Sec. 53.

Solution. By the use of the graph given in Fig. 7.31 determine h_{opt} for the section modulus $W_x = 1{,}125$ cm³. We obtain $h_{opt} = 48$ cm. The cross-sectional area according to formula (7.35) is

$$F = \frac{2 \times 1{,}125}{48} + \frac{2}{3} 0.2 \times 48 + \frac{2}{3} 0.0129 \times 48^2 = 73 \text{ cm}^2.$$

In the example of Sec. 53 we chose I-beam No. 45 whose cross-sectional area is $F = 83.0$ cm², which is 14% greater than the optimum cross-sectional area (based on strength).

According to the standard of 1939 (OST 10016-39) we should take I-beam No. 40*b* for which $W_x = 1{,}140$ cm³, $F = 94.1$ cm².

As can be seen, the standard of 1956 (GOST 8239-56) is more economical than the standard of 1939.

CHAPTER VIII

STRENGTH THEORIES

68. Object of Strength Theories

We have so far been concerned with strength analysis in cases when the material undergoes either a uniaxial state of stress (tension, compression) or the simplest biaxial state of stress when the principal stresses at each point are equal in magnitude and opposite in sign (pure shear which takes place in torsion).

The derivation of the conditions of strength in these cases offers no difficulty. To ensure strength in a material it is necessary that the maximum normal stress (in tension or compression) or the maximum shearing stress (in torsion) shall not exceed the appropriate allowable stress, the value of which is established on the basis of the corresponding yield point or ultimate strength (for brittle materials) obtained experimentally.

Henceforth, in the discussion of combined loadings such as torsion with bending, etc., more complicated stress conditions will be encountered.

Fig. 8.1 shows the general case of the triaxial state of stress with the plane of the maximum shearing stress shaded. It is well to recall that the following rule has been used previously to designate the principal stresses: $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ (with regard to sign).

The question is at what values of the stresses σ_{1l}, σ_{2l}, σ_{3l} the limiting state of the material will take place, i.e., at what instant fracture will occur or yielding start.

The solution of this question would make it possible to solve another problem: determine the safe allowable values of the principal stresses σ_1, σ_2, and σ_3.

The problem stated is very complicated. The most reliable way of solving this problem would be to test a specimen for a given relation between the principal stresses

up to fracture or the onset of yielding and to establish thus the ultimate and then the allowable values of the principal stresses.

However, this method has to be rejected, since for each new combination of stresses it would be necessary to carry out the test anew.

Furthermore, the performance of such experiments requires very complex machines and apparatus. It is desirable, therefore, to have some theory which will predict the likelihood of the transition of a material to the limiting state under combined stress without resorting each time to complex tests but using only data of the simplest tests, i.e., tests with the uniaxial state of stress.

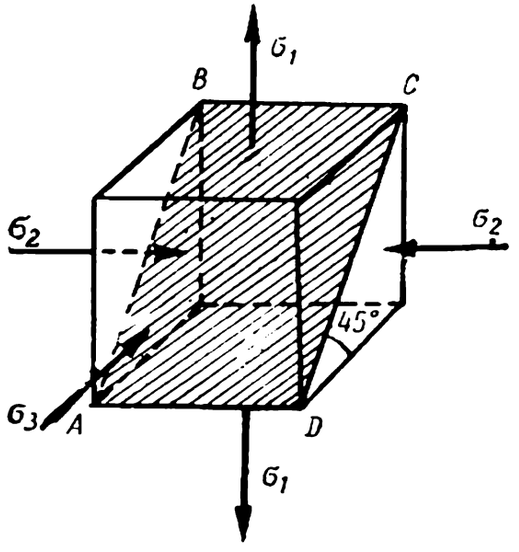


Fig. 8.1

Several such theories have been proposed and investigations in this field are continuing. This is due to the complex nature of failure. From the physical point of view, failure of a material represents either direct separation of particles from each other (so-called brittle fracture) or slipping of particles (so-called ductile failure accompanied by considerable plastic deformations).

However, the complexity of the problem consists in the fact that one and the same material may fracture both in a brittle and ductile manner under different stress conditions and different testing conditions (temperature of the surrounding medium, rate of strain, etc.). Moreover, in certain cases a combined type of failure is possible when in some zones failure will occur by direct separation of particles and in other zones by slip. This indicates that the nature of the limiting state of a material and the conditions of its transition to the limiting state depend on many factors.

It is natural to take stresses (normal and shearing) and deformations (linear and angular) as such factors. It has also been proposed that the potential energy of strain be taken as the criterion of the transition to the limiting

state. The idea of the *theories (hypotheses) of strength* to be discussed below is that each of them chooses any one of a great number of factors influencing the strength of a material and disregards all the others (we emphasise that here and subsequently when speaking of strength we imply both actual fracture and the onset of yielding).

With the accumulation of test data it became apparent that application of more complex theories of strength was needed, based not only on tests with uniaxial tension (compression) but also on tests under combined stress.

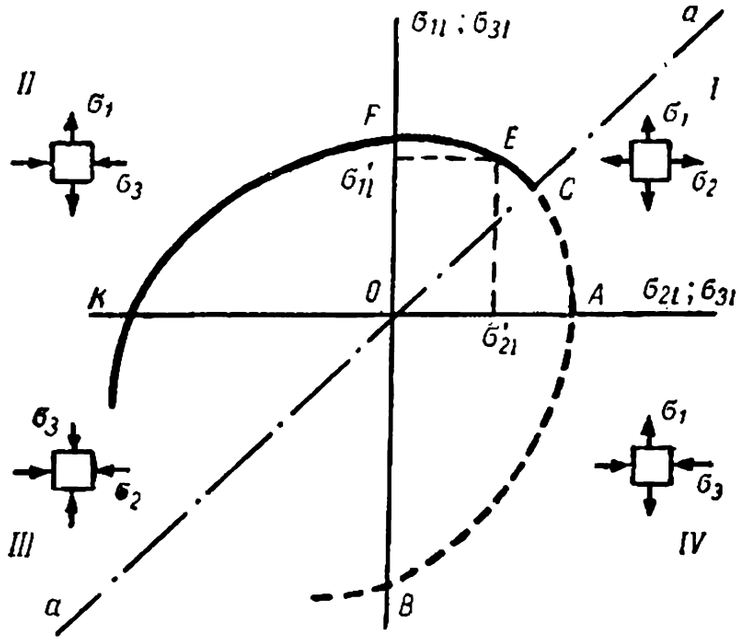


Fig. 8.2

The reliability of one or another theory of strength is verified experimentally. Therefore, before turning to the treatment of strength theories let us look at the results of some tests carried out under a biaxial state of stress. Biaxial states of stress with different ratios of the principal stresses are comparatively simple to obtain in tests with thin-walled tubes subjected to an internal pressure and axial force.

The results of tests will be represented graphically (Fig. 8.2) by plotting the relation between the values of the principal stresses by the moment of fracture or by the moment of the onset of yielding of the material (in short, the relation between the ultimate values of the principal stresses).

Then the ultimate stress in uniaxial tension will be represented by segment *OF* (if the element shown in the drawing is extended in a vertical direction) or *OA* (in case of extension of the element in a horizontal direction); for isotropic materials $OF=OA$. The magnitude of these segments is equal either to the ultimate strength or to the yield point stress in uniaxial tension according to the limiting state under consideration. *

In uniaxial compression the ultimate stress will be represented by segment *OB* or *OK*.

Let us now test a specimen under any biaxial state of stress, for instance, one in which stress σ_1 while increasing is always twice as large as stress σ_2. At certain values of these stresses, for example, σ'_{1l} and σ'_{2l} fracture will occur or the material will start to yield. Plot point *E* whose co-ordinates are σ'_{1l} and σ'_{2l}. Carrying out tests for other relations between the principal stresses, and plotting the corresponding points and joining them we obtain a certain line *KFCAB* which will be called the *ultimate stress diagram.* It is evident that for isotropic materials the line *a-a* is the axis of symmetry of this diagram so that it is sufficient to plot one half of the ultimate stress diagram: *CEFK* or *CAB* [cases of biaxial compression (quadrant *III* of the diagram) are excluded from consideration].

The points lying in the first quadrant of the diagram will characterise biaxial tension ($\sigma_3=0$, $\sigma_1>0$ and $\sigma_2>0$); the points lying in the second and fourth quadrants—biaxial tension-compression ($\sigma_1>0$, $\sigma_2=0$, $\sigma_3<0$); the points lying in the third quadrant—biaxial compression ($\sigma_2<0$, $\sigma_3<0$, $\sigma_1=0$).

Fig. 8.3 gives the ultimate stress diagram for a brittle material: grey cast iron. The chemical composition of the

---

* In constructing this diagram the rule for designating the principal stresses is partially violated. For instance, for quadrant *I* the abscissas of points in any biaxial tension are equal to the ultimate value of the principal stress acting horizontally and designated by σ_2 (see the representation of an element being stretched in quadrant *I*) though for the points lying below the straight line *OC* $\sigma_2>\sigma_1$, i.e., if the rule of designations be strictly observed, the stress acting horizontally should have been considered the first principal stress.

cast iron is: C—3.48%, Si—2.41%, Mn—0.52%. Tests were carried out on cast-iron tubes having an outside diameter of 14 mm and a wall thickness of 0.75 mm. The tubes were subjected to combined axial load and internal pressure.

It is seen from the diagram that in the first quadrant fracture occurs at a practically constant tensile stress of 19 kg/mm². In the fourth quadrant the tensile stress remains constant (19 kg/mm²) until the second (now compressive) stress exceeds 23 kg/mm². After this the tensile stress begins to decrease while the compressive stress grows, reaching 63 kg/mm² (the ultimate strength in uniaxial compression). The third quadrant has not been investigated.

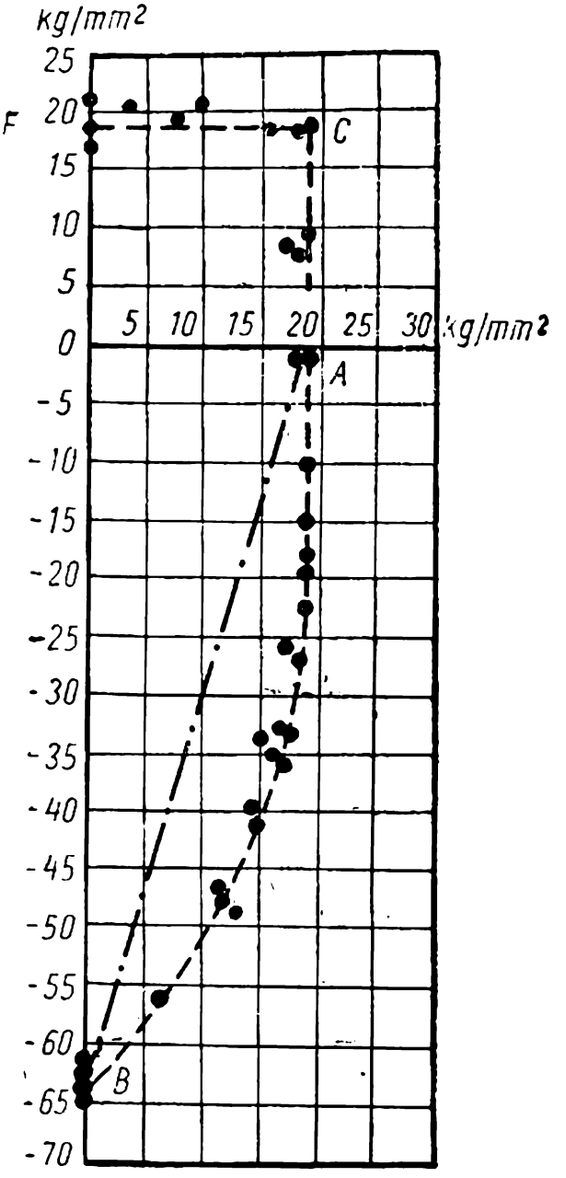


Fig. 8.3

In Fig. 8.4 are shown the ultimate stress diagrams for other highly brittle materials: glass (Fig. 8.4*a*) and plaster (Fig. 8.4*b*).

Glass specimens were made in the form of tubes 220 mm long with an inside diameter of 35 mm and a wall thickness of 1 mm. Plaster specimens were cast from medical plaster also in the form of tubes with an inside diameter of 39.5 mm and a wall thickness of 3 mm. For glass intermediate points between $\sigma_3 = -30$ kg/mm² and $\sigma_3 = -87.5$ kg/mm² could not be obtained because of the insufficient capacity of the testing machine.

From Fig. 8.4*a* it is seen that for glass up to $\sigma_3 = -30$ kg/mm² the tensile stress $\sigma_1 = +4$ kg/mm² plays a decisive part. For plaster up to $\sigma_3 = -1.2$ kg/mm² the tensile stress $\sigma_1 = +0.4$ kg/mm² also plays a decisive part.

The third quadrant (i.e., cases of biaxial compression) has not been investigated.

Fig. 8.5 shows the results of tests for steel and copper, which belong to the group of ductile materials.

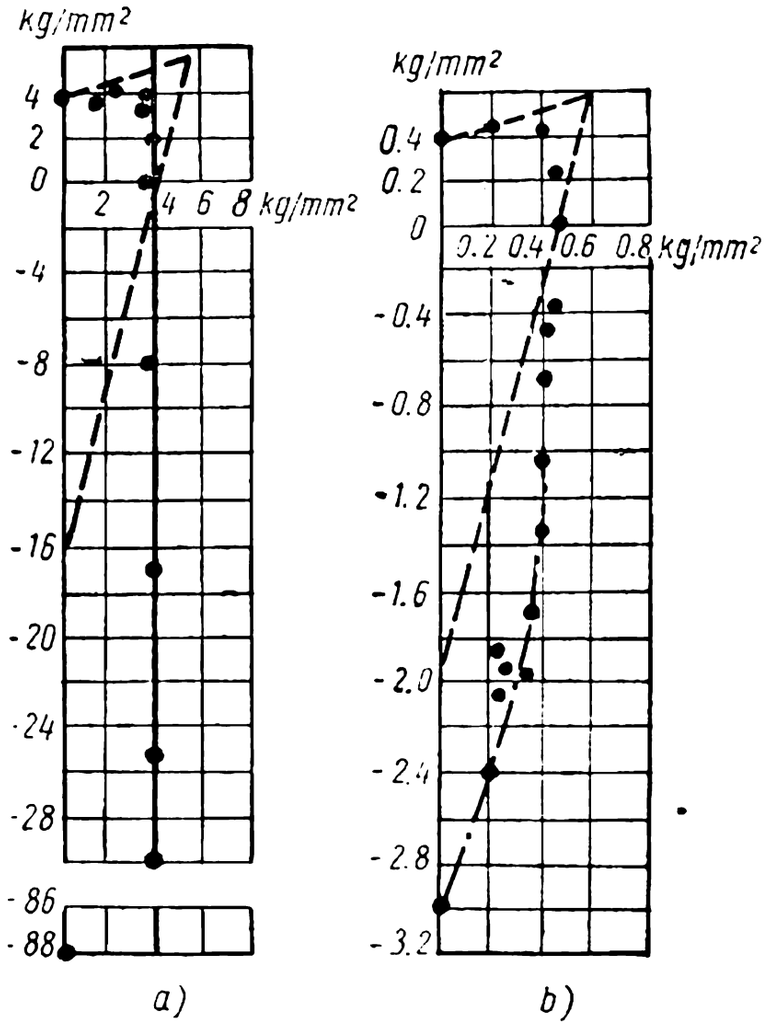


Fig. 8.4

Here the ratios of the ultimate stresses to the yield point in uniaxial tension (compression), rather than their absolute values, are laid off along the co-ordinate axes. From this diagram it is seen that the test points lie on a certain curve *CKADB*. This indicates that here, unlike the preceding diagrams, neither of the stresses plays a dominating part.

For the triaxial state of stress there are considerably less test data than for the plane state of stress. The tests have been carried out only for a certain limited number of combinations of the principal stresses.

Let us turn now to the treatment of the basic strength theories available. Some of them are confirmed by experiments simultaneously as theories determining the resist-

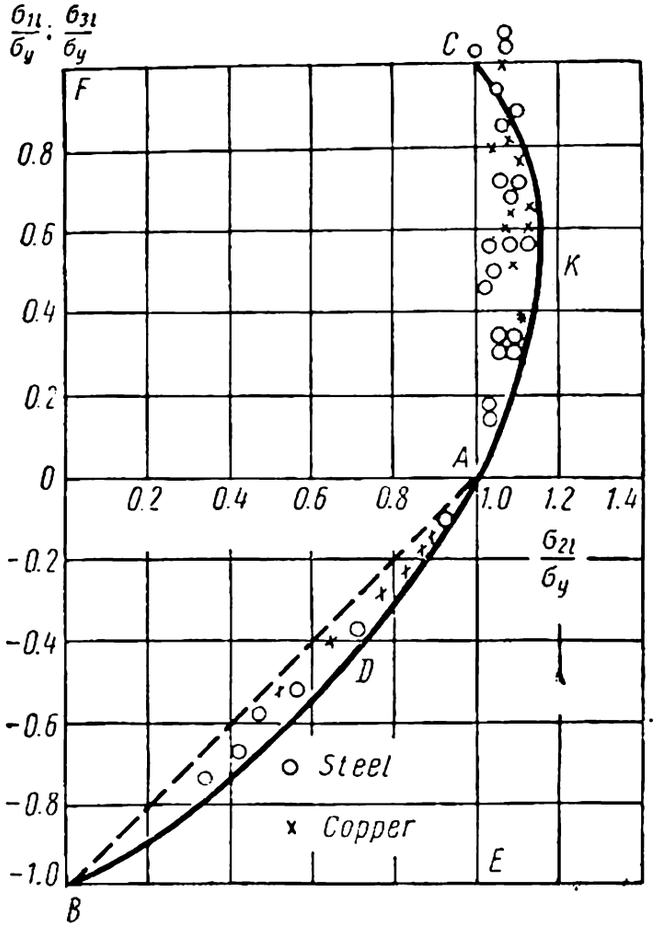


Fig. 8.5

ance of a material to failure and as theories determining the resistance of a material to plastic deformations (the onset of yielding).

69. First Theory of Strength

The first theory of strength is also called *the maximum normal stress theory*, because it takes the maximum normal stress as the criterion for strength. It can be formulated as follows.

The limiting state of a material subjected to combined stress occurs when the maximum normal stress attains the magnitude of the ultimate stress in the uniaxial state of stress:

$$\sigma_{\max l} = \sigma_l. \tag{8.1}$$

Here $\sigma_{\max_l}$ is the ultimate value of the maximum (of the three) normal stress (at the moment of fracture or the beginning of yielding of the material); σ_l is the ultimate stress in uniaxial compression or tension (the ultimate strength or yield point stress).

The ultimate stress diagram based on the first theory is shown in Fig. 8.6*a* in the form of the straight lines 1-2, 2-3, 3-4 and 1-4. Segments $OA=OF$ represent the ultimate stress in uniaxial tension, and segments $OB=OK$ repre-

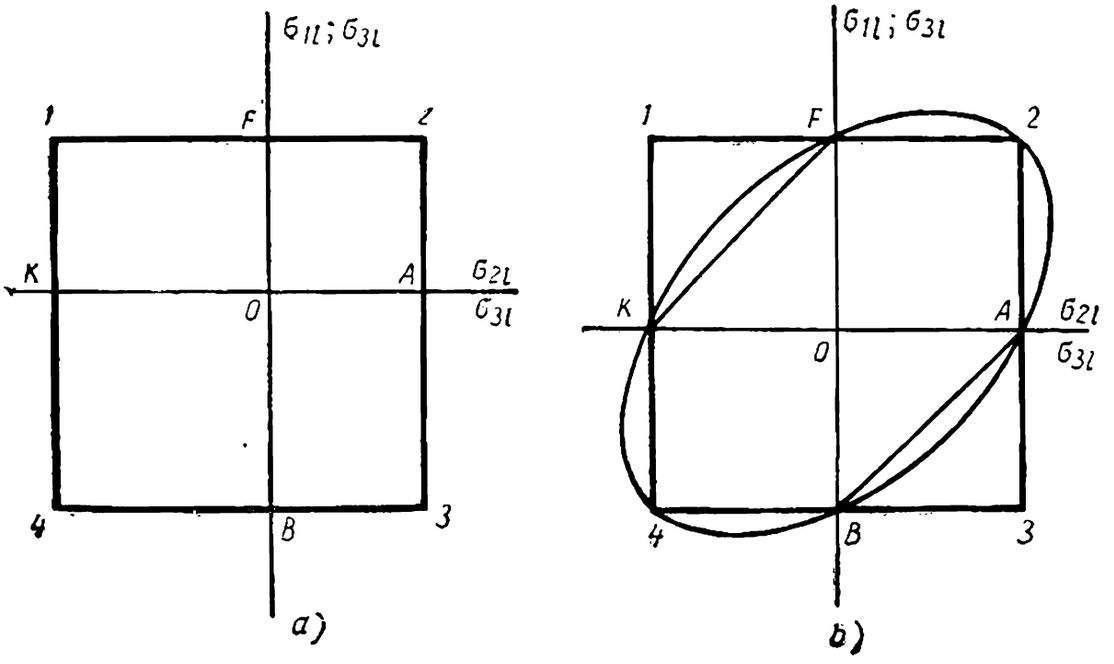


Fig. 8.6

sent the ultimate stress in uniaxial compression. For brittle materials $OB>OF$; for ductile materials $OB=OF$ (Fig. 8.6*b*). Dividing both members of equality (8.1) by the factor of safety we obtain

$$\sigma_{\max} \leqslant [\sigma], \tag{8.2}$$

where $\sigma_{\max}$ is the actual value of the maximum (in absolute value) principal stress occurring at the dangerous point of a part; $[\sigma]$ is the allowable value of the normal stress as assumed for uniaxial tension or compression.

Based on inequality (8.2) the first theory of strength can also be formulated as follows.

A material will not fail under combined stress if the maximum normal stress does not exceed the allowable normal stress assumed for the uniaxial state of stress.

As is seen, this theory takes into consideration only the influence of the maximum (in absolute value) principal stress, disregarding the effect of the other two principal stresses on the strength of a material. The foregoing experiments for ductile materials do not confirm this hypothesis as a hypothesis of plasticity, i.e., a hypothesis establishing the criterion for the onset of yielding (see Fig. 8.5). The test points lie on a certain curve *CKADB* but not on the straight lines *FC*, *CE*, and *BE*, as follows from the first theory.

For brittle materials in the first quadrant of the ultimate stress diagram and in a considerable part of the fourth (or the second) quadrant the first theory is in full agreement with test data (see Figs. 8.3 and 8.4). In these cases it is this theory that should be used, being looked upon as a hypothesis establishing the criterion for brittle fracture.

70. Second Theory of Strength

The second theory of strength is also called *the maximum linear strain theory.* According to this theory *a material will not fail under combined stress if the maximum unit linear strain does not exceed the allowable unit linear strain assumed for uniaxial tension.*

Thus, the condition of strength based on the second theory is

$$\varepsilon_{max} \leqslant [\varepsilon], \tag{8.3}$$

where $[\varepsilon]$ is the allowable unit strain for uniaxial tension.

Assuming that the material obeys Hooke's law up to failure (a permissible assumption only for brittle materials), we can replace strains by stresses, using generalised Hooke's law (see Sec. 20).

In accordance with the rule adopted for designating the principal stresses ($\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$) the maximum in the algebraic sense of the three strains ε_1, ε_2 and ε_3 will be ε_1:

$$\varepsilon_{max} = \varepsilon_1 = \sigma_1 - \mu(\sigma_2 + \sigma_3). \tag{8.4}$$

The allowable strain $[\varepsilon]$ is determined from uniaxial tension tests.

If the allowable value for stresses is $[\sigma]$, on the basis of Hooke's law the allowable value for strains is thereby

$$[\varepsilon] = \frac{[\sigma]}{E}. \tag{8.5}$$

Bearing this in mind, condition (8.3) can be expressed in terms of stresses according to formula (8.4) as

$$\sigma_{eq} = \sigma_1 - \mu(\sigma_2 + \sigma_3) \leqslant [\sigma]. \tag{8.6}$$

The expression in the left-hand member of inequality (8.6) represents a certain stress called the *reduced*, or *equivalent*, stress.

The second theory is not confirmed by experimental data. In reality, according to this theory a specimen subjected to tension in two directions should sustain greater stress than in simple tension. This conclusion, however, is not supported by experiments.

71. Third Theory of Strength

According to the third theory of strength, also called *the maximum shearing stress theory, a material will not fail under combined stress if the maximum shearing stress does not exceed the allowable shearing stress established from tests with the uniaxial state of stress*:

$$\tau_{max} \leqslant [\tau]. \tag{8.7}$$

It was shown in Sec. 18 that under a biaxial state of stress the maximum shearing stresses occur in sections at an angle $\alpha = 45°$ to the direction of the principal stresses and are equal to half the difference between these stresses. In the case of a three-dimensional state of stress the maximum shearing stresses occur on three diagonal planes one of which is shown in Fig. 8.1, their magnitude on the *ABCD* plane being

$$\tau'_{max} = \frac{\sigma_1 - \sigma_3}{2}.$$

Similarly, on two other planes

$$\tau''_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

and

$$\tau'''_{max} = \frac{\sigma_2 - \sigma_3}{2}.$$

The first of these three quantities will always be greater than the other two because of the rule adopted for subscripts ($\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$).

The allowable shearing stress $[\tau]$ under a uniaxial state of stress is related to the allowable normal stress $[\sigma]$ by the formula

$$[\tau] = \frac{[\sigma]}{2},$$

which follows from the above formulas if σ_2 and σ_3 are set equal to zero in them.

Thus, the condition of strength according to the third theory, expressed in terms of normal stresses, is

$$\sigma_{eq} = \sigma_1 - \sigma_3 \leqslant [\sigma]. \tag{8.8}$$

In order to compare this theory with experimental data let us also write down the condition of transition of a material to the limiting state according to the third theory:

$$\sigma_{1l} - \sigma_{3l} = \sigma_l. \tag{8.9}$$

Here σ_{1l} and σ_{3l} are the values of the principal stresses in the limiting state; σ_l is the ultimate stress in uniaxial compression or tension.

Represent condition (8.9) on the ultimate stress diagram. In biaxial tension the minimum stress σ_3 is zero, therefore in the first quadrant, as well as in the third, the third theory coincides with the first theory—the lines $F2$, $2A$, $K4$ and $4B$ (see Fig. 8.6*b*).

In the second quadrant equation (8.9) represents the equation of line KF inclined at an angle of 45° (see Fig. 8.6*b*). In the fourth quadrant line AB corresponds to line KF.

Comparing the ultimate stress diagram based on the third theory (straight lines CA and AB in Fig. 8.5) with the results of tests for ductile materials, we see that in general the third theory characterises the resistance of these materials to plastic deformations satisfactorily, in any case much more correctly than the first theory (lines CE and BE).

For a case of pure shear (torsion), when $\sigma_1 = \tau$ and $\sigma_3 = -\tau$ according to formula (8.9) we obtain

$$2\tau_y = \sigma_y$$

or

$$\tau_y = 0.5\sigma_y.$$

Here τ_y is the yield point in pure shear, σ_y is the yield point in uniaxial tension (compression).

Experiments give the value $\tau_y \approx 0.6\sigma_y$ for most steels.

The shortcoming of the third theory is that it takes into consideration neither the effect of the intermediate principal stress (σ_2) nor the different resistance of a material in tension and compression.

At present the third theory of strength is widely used for ductile materials having the same strength in tension and compression.

For brittle materials the third theory gives good results only for the first quadrant of the ultimate stress diagram (biaxial tension) where it coincides with the first theory.

72. Energy Theories of Strength

According to the first of the energy theories *a material will not fail under combined stress if the potential strain energy per unit of volume does not exceed the allowable potential energy established from tests with the uniaxial state of stress*:

$$u \leqslant [u]. \tag{8.10}$$

The unit potential strain energy in a three-dimensional state of stress is (see Sec. 21)

$$u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right]. \tag{8.11}$$

This quantity is always positive. Therefore, the energy theory, like the third theory, does not take into account the difference between tension and compression; in other words, when using this theory one must assume that

$$[\sigma_t] = [\sigma_c] = [\sigma].$$

The allowable unit potential energy in the uniaxial state of stress when $\sigma_1 = [\sigma]$, $\sigma_2 = \sigma_3 = 0$ according to the same formula (8.11) is given by

$$[u] = \frac{[\sigma]^2}{2E}. \tag{8.12}$$

Substituting the values of u from formula (8.11) and $[u]$ from formula (8.12) in formula (8.10) we obtain

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) \leqslant [\sigma]^2$$

or

$$\sigma_{eq} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)} \leqslant [\sigma]. \tag{8.13}$$

Better results are obtained if only that part of the strain energy which is due to a change in the shape of a body is taken as the criterion for strength rather than the total strain energy.

This condition can most easily be written down assuming $\mu=0.5$ in formula (8.13), since at $\mu=0.5$ the volume of a body does not change (see Sec. 20).

Consequently, the condition of strength (in this case it is the yield condition) based on the distortion energy theory (also called the fourth theory) is

$$\sigma_{eq}=\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2-\sigma_1\sigma_2-\sigma_2\sigma_3-\sigma_3\sigma_1}\leqslant[\sigma]. \quad (8.14)$$

In the special case of plane stress ($\sigma_3=0$) we obtain

$$\sigma_{eq}=\sqrt{\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2}\leqslant[\sigma]. \quad (8.15)$$

For comparison with test data let us write down the limiting condition according to this theory for plane stress:

$$\sigma_{1l}^2+\sigma_{2l}^2-\sigma_{1l}\sigma_{2l}=\sigma_l^2. \quad (8.16)$$

Here σ_{1l} and σ_{2l} are the ultimate values of the principal stresses, σ_l is the ultimate value of the stress in uniaxial tension (compression).

This equation is represented on the ultimate stress diagram by an ellipse (Fig. 8.6*b*). Comparing the yield condition based on the fourth theory (the line *CKADB* in Fig. 8.5) with test data, we find a very close coincidence. For a case of pure shear we obtain from (8.16) $3\tau_y^2=\sigma_y^2$, from where $\tau_y\approx0.58\,\sigma_y$ which is in good agreement with test results. Experiments confirm this hypothesis not only as a hypothesis of plasticity but also as a hypothesis of strength for ductile materials having the same resistance to tension and compression.

For brittle materials the results based on the fourth theory are found unsatisfactory.

73. Trends in Development of Strength Theories

The foregoing theories of strength give fair results either only for brittle fracture (the first theory) or only for ductile failure (the third or fourth theory). Furthermore, they do not take into account different strengths of materials in tension and compression.

A great number of other theories have been proposed besides those considered here. Some of them are worth attention though they cannot be treated in detail in the present concise course. Of particular note is Mohr's theory of strength, which is a development of the third theory and with which it is possible to allow for differences in the resistance of materials to tension and compression. The condition of strength based on Mohr's theory is

$$\sigma_{eq} = \sigma_1 - \nu\sigma_3 \leqslant [\sigma_t], \tag{8.17}$$

where $\nu = \frac{[\sigma_t]}{[\sigma_c]}$, $[\sigma_t]$ is the allowable stress in tension, $[\sigma_c]$ is the allowable stress in compression.

For $\nu = 1$ Mohr's theory of strength coincides with the third theory. On the ultimate stress diagram Mohr's theory coincides with the first and third theories of strength (lines *FC* and *CA* in Fig. 8.3) in the first quadrant, and in the fourth quadrant it gives the relation between the ultimate values of stresses σ_{1l} and σ_{3l} in the form of straight line *AB*. As can be observed, Mohr's theory gives satisfactory results for brittle materials.

Note must also be taken of the generalisation of the energy theory of strength proposed by P. P. Balandin, with which different strengths of materials in tension and compression can be taken into account. The condition of strength according to P. P. Balandin is

$$\sigma_{eq} = \frac{1-\nu}{2}(\sigma_1 + \sigma_2 + \sigma_3) + \\ + \frac{1}{2}\sqrt{(1-\nu)^2 + (\sigma_1+\sigma_2+\sigma_3)^2 + 4\nu\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right]} \leqslant \\ \leqslant [\sigma_t],$$

where

$$\nu = \frac{[\sigma_t]}{[\sigma_c]}.$$

For a biaxial state of stress at $\sigma_3 = 0$ we obtain

$$\sigma_{eq} = \frac{1-\nu}{2}(\sigma_1 + \sigma_2) + \\ + \frac{1}{2}\sqrt{(1-\nu)^2(\sigma_1+\sigma_2)^2 + 4\nu(\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2)} \leqslant [\sigma_t].$$

For materials having the same resistance to tension and compression, i.e., at $\nu = 1$, Balandin's theory coincides with the fourth theory of strength.

Experiments show that Balandin's theory gives satisfactory results over a certain range of stress conditions. Thus, for cast iron it can be applied in the second part of the fourth quadrant of stresses (see Fig. 8.3) from the moment when the compressive stress becomes equal to the tensile stress, and apparently in the third quadrant as well (though experimental data for the third quadrant are still scarce).

Attention is also due to the so-called *combined theories of strength* advanced lately in which two or even three factors are taken as the criterion for strength rather than one factor. Professor Y. B. Fridman proposed a combination of the second and third theories of strength. Academician N. N. Davidenkov proposed a combination of the first theory and P. P. Balandin's theory on the basis of the tests referred to above with cast iron, glass and plaster.

For instance, for cast iron the first theory must be applied in the first quadrant of stresses and in the first part of the fourth quadrant until the compressive stress exceeds the tensile stress in magnitude.

In the remaining part of the fourth (or the second) quadrant, and also in the third quadrant of stresses, Academician N. N. Davidenkov considers P. P. Balandin's theory of strength to be applicable.

Other theories of strength have also been advanced. Professor M. M. Filonenko-Borodich proposed that the condition of strength be written as a polynomial of the second or even third degree with respect to the principal stresses and containing a certain number of arbitrary constants to be determined by tests, including tests under combined stress. However, the above diagrams of fracture of brittle materials indicate clearly that the condition of strength of a material cannot be expressed by a continuous function over the entire range of stress conditions.

For brittle materials N. N. Davidenkov's combined theory of strength seems to be most justifiable.

We have discussed the strength theories on the basis of results of tests under a biaxial state of stress. There are considerably less test data regarding triaxial states of stress. The experiments carried out indicate that under stress conditions close to triaxial compression, materials, even brittle ones, are capable of sustaining very high

stresses. When materials such as steel, copper or aluminium are subjected to equal triaxial compression no fracture occurs under enormous pressures of 50,000-200,000 kg/cm².

On the basis of the test data available it can be considered that for ductile materials under a triaxial state of stress the distortion energy theory and the third theory of strength give satisfactory results. As regards brittle materials, the first theory of strength is recommended for cases where all three principal stresses are positive (tensile). For other states of stress many authors recommend Mohr's theory of strength.

Example 8.1. Using the fourth theory of strength, check the strength of a material (steel 20) subjected to triaxial compression $\sigma_3 = -5{,}000$ kg/cm²; $\sigma_2 = -4{,}000$ kg/cm²; $\sigma_1 = -4{,}000$ kg/cm²; $[\sigma_t] = 1{,}600$ kg/cm².

Solution. According to the distortion energy theory the equivalent stress will be (see formula 8.14)

$$\sigma_{eq} = \sqrt{\sigma_3^2 + \sigma_2^2 + \sigma_1^2 - \sigma_3\sigma_1 - \sigma_2\sigma_1 - \sigma_2\sigma_3} =$$

$$= \sqrt{5{,}000^2 + 4{,}000^2 + 4{,}000^2 - 5{,}000 \times 4{,}000 - 4{,}000 \times 4{,}000 - 4{,}000 \times 5{,}000} =$$

$$= 1{,}000 \text{ kg/cm}^2 < 1{,}600 \text{ kg/cm}^2$$

It is of interest to note that according to the first theory of strength, which is not applicable in this case, we would have $\sigma_{eq} = 5{,}000$ kg/cm², i.e., considerably higher.

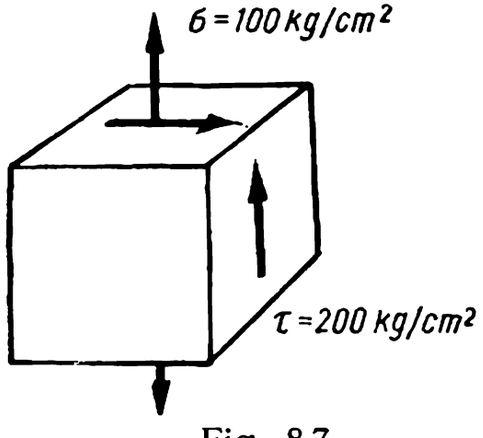


Fig. 8.7

Example 8.2. Check the strength of a cast-iron part the state of stress at the dangerous point of which is shown in Fig. 8.7. The allowable stresses are: $[\sigma_t] = 350$ kg/cm², $[\sigma_c] = 1{,}200$ kg/cm², $\nu = 0.29$.

Solution. Since the state of stress is given in terms of non-principal stresses we calculate first the principal stresses by the use of formulas in Sec. 19:

$$\sigma_1 = \frac{\sigma}{2} + \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2} = \frac{100}{2} + \frac{1}{2}\sqrt{100^2 + 4 \times 200^2} =$$

$$= 50 + 206 = 256 \text{ kg/cm}^2,$$

$$\sigma_2 = \frac{\sigma}{2} - \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2} = 50 - 206 = -156 \text{ kg/cm}^2,$$

Since the tensile stress is numerically larger than the compressive stress, according to N. N. Davidenkov's combined theory of strength the maximum normal stress theory should be applied to this state of stress:

$$\sigma_{eq} = 256 \text{ kg/cm}^2 < 350 \text{ kg/cm}^2.$$

Consequently, failure will not occur.

Example 8.3. Calculate the factor of safety for a cast-iron part if the principal stresses at the dangerous point are: $\sigma_1 = 250$ kg/cm^2, $\sigma_2 = 0$, $\sigma_3 = -500$ kg/cm^2. The ultimate strength of the cast iron in tension is $\sigma_{u.t} = 1{,}600$ kg/cm^2, in compression $\sigma_{u.c} = 5{,}500$ kg/cm^2, $\nu = 0.29$.

Solution. Since the compressive stress exceeds the tensile one in magnitude, the equivalent stress is calculated according to P. P. Balandin's theory of strength:

$$\sigma_{eq} = \frac{1-\nu}{2}(\sigma_1 + \sigma_3) +$$

$$+ \frac{1}{2}\sqrt{(1-\nu)^2(\sigma_1+\sigma_3)^2 + 4\nu(\sigma_1^2+\sigma_3^2-\sigma_1\sigma_3)} =$$

$$= \frac{1-0.29}{2}(250-500) +$$

$$+ \frac{1}{2}\sqrt{(1-0.29)^2(250-500)^2 + 4\times 0.29\,(250^2+500^2+250\times 500)} =$$

$$= 276.5 \text{ kg/cm}^2.$$

The safety factor is

$$n = \frac{1{,}600}{276.5} = 5.8.$$

For the sake of comparison calculate the factor of safety according to Mohr's theory:

$$\sigma_{eq} = \sigma_1 - \nu\sigma_3 = 250 + 0.29 \times 500 = 395 \text{ kg/cm}^2,$$

$$n = \frac{1{,}600}{395} = 4.06.$$

CHAPTER IX

COMPLEX DEFORMATIONS (COMPLEX RESISTANCE)

74. Basic Concepts

We have studied four types of simple loading of a bar: central tension (compression), shear, torsion, plane (direct) bending.

In all these cases only one internal force (longitudinal or transverse force, twisting or bending moment) occurred at cross sections of a bar under the action of a load. The general case of plane bending (transverse bending) was the only exception in which two internal forces, bending moment and shearing force, were produced simultaneously at cross sections of a bar. But even in this case, in the analysis of strength and rigidity only one internal force, generally the bending moment, was as a rule taken into consideration.

However, more complicated cases are often encountered in practice in which several internal force factors (internal forces) act simultaneously on cross sections of a bar, all of them being taken into account in the strength analysis, for instance, longitudinal force and twisting moment, or a combination of three and more internal forces. Such cases are called *complex deformations.* *

The procedure for solving such problems is as follows.

At first internal force factors occurring at cross sections of a bar are determined by means of the method of sections.

Under combined loading it is recommended to construct the internal force diagrams enabling the dangerous section to be located. ** After that, on the basis of the principle

---

* Occasionally called complex resistance.

** In some cases it is not possible from internal force diagrams to establish with certainty which section is dangerous; then two (and sometimes even more) presumably dangerous sections are found according to the diagrams and an analysis is made for each of them.

of superposition, normal and shearing stresses due to each internal force are determined separately by the use of formulas obtained in the preceding chapters. By analysing the stress distribution over the section the dangerous (or presumably dangerous) point is established for which the condition of strength is derived. If it turns out that the uniaxial state of stress (uniaxial tension or compression) takes place at the dangerous point, then for the strength analysis it is sufficient to compare the total (i.e., due to all internal forces) normal stress occurring at this point with the allowable stress ($[\sigma_t]$ or $[\sigma_c]$). In a case where the state of stress at the dangerous point is biaxial *, the analysis should be carried out by applying one or another theory of strength. As is known from the above, the choice of the theory of strength is determined primarily by the material behaviour (ductile or brittle).

The principle of superposition is also used to determine displacements, if necessary.

75. Examples of Constructing Internal Force Diagrams for Bar with Broken Axis

Consider an example of constructing the twisting and bending moment diagrams and the longitudinal force diagram for the bar with a broken axis represented in Fig. 9.1*a*. The system of co-ordinates is shown in the same figure.

The bending moment at any cross section of the bar is determined as the algebraic sum of the moments (with respect to the corresponding axis) of the external forces acting on one side of the cross section. In order not to determine the reaction at the built-in end in advance it is recommended to take the sum of the moments of the forces lying from the side of the free end of the bar.

The bending moment at any cross section of portion AB is $M_{u_1} = Pu_1$ ** (Fig. 9.1*b*).

According to this equation the M_b diagram is constructed for the portion AB on the right side of the bar (on the

* In the design of a bar cases of triaxial stress are not encountered.

** The distances from one of the ends of the given portion of the bar to the cross section being considered are designated by u_1, u_2, and u_3 respectively.

compressed fibre). To determine the bending moments at sections of portion BC it is advisable to transmit force P parallel to itself from point A to point B. In transmitting the force it is necessary to add a moment lying in the plane of the drawing, i.e., in the ZOY plane, and equal to $M_{yz}=Pl_1$. Since the plane of action of this moment is perpendicular to the axis of the portion BC of the bar it

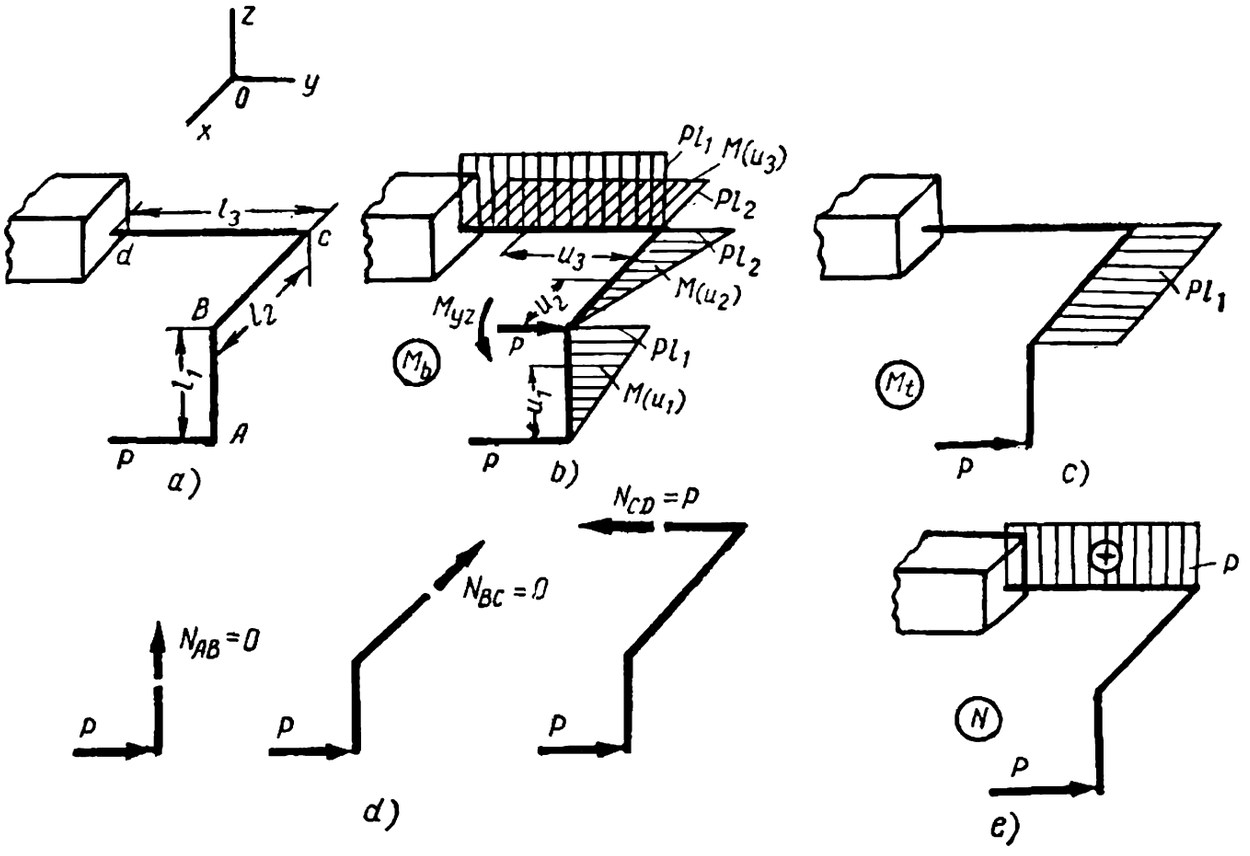


Fig. 9.1

causes torsion, while bending is produced by force P only. Therefore the bending moment at the cross section having the abscissa u_2 is

$$M_{u_2}=Pu_2.$$

The compressed fibres in portion BC will be to the right and it is here that the M_{u_2} diagram with the maximum ordinate Pl_2 is constructed. In a similar way the bending moment diagram is constructed for portion CD. For that force P and moment $M_{yz}=Pl_1$ are transmitted from point B to point C.

Acting at point C will be force P directed along portion CD and two moments: the moment $M_{yz}=Pl_1$ which is

transmitted unchanged and the second moment $M_{xy}=Pl_2$ which is due to the transmission of force P from point B to point C.

The two moments M_{yz} and M_{xy} (they are not indicated in the drawing) produce bending: the first moment in the vertical plane and the second in the horizontal plane.

The corresponding bending moment diagrams for portion CD are shown in Fig. 9.1*b*. The diagrams are constructed on the side of the compressed fibres of the portion under consideration.

Let us construct the twisting moment diagram. The bar in the portion AB is not subjected to torsion since force P lies in the same plane as the longitudinal axis of the portion AB.

The twisting moment at cross sections of the portion BC is

$$M_t = M_{yz} = Pl_1.$$

The M_t diagram may be constructed on either side of the bar (Fig. 9.1*c*).

The portion CD is not subjected to torsion since force P is parallel to its axis.

Next construct the longitudinal force diagram. From the condition of equilibrium of the cut parts we obtain by projecting the forces on the directions of the axes of the individual portions of the bar (Fig. 9.1*d*),

$$N_{AB}=0, \quad N_{BC}=0, \quad N_{CD}=P \text{ (tension)}.$$

The N diagram is shown in Fig. 9.1*e*.

Analysing the diagrams we see that

(1) portion AB is under bending only,

(2) portion BC is under torsion and bending in the horizontal plane,

(3) portion CD is under bending in two planes and also tension.

The strength analysis of each portion of the bar must be carried out accordingly.

76. Bending in Two Planes (Oblique Bending)

Oblique bending occurs in cases where external forces perpendicular to the axis of a bar do not lie in a plane passing through a principal axis of inertia of its cross

section (Fig. 9.2). In this case the bending moment produced in the cross section can be resolved into two bending moments acting in the planes passing through the principal axes of inertia of the cross section. Thus, oblique bending may be considered as a combination of two plane bendings in mutually perpendicular planes.

In oblique bending the normal stress at any point on the cross section is equal to the algebraic sum of the stresses produced by the bending moments acting in the two planes.

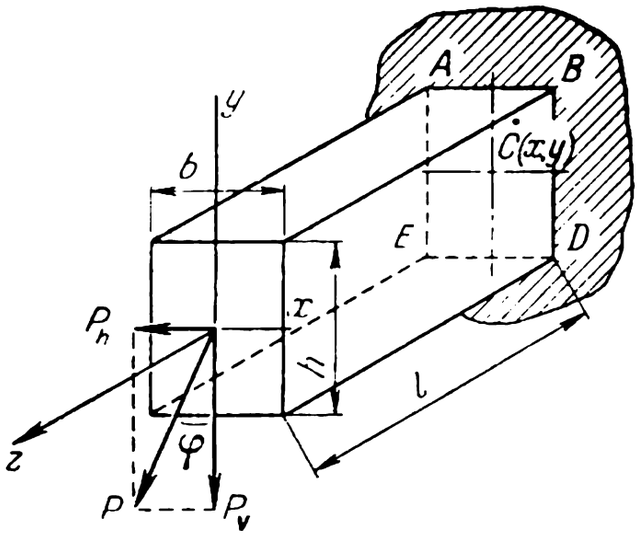


Fig. 9.2

Consider, for instance, a point C on the section over the support having the co-ordinates x and y with respect to the principal centroidal axes.

At this section

(a) the bending moment producing the bending of the bar in the vertical plane with the neutral x-axis is

$$M_x = P_v l = P \cos \varphi l;$$

(b) the bending moment producing the bending of the bar in the horizontal plane with the neutral y-axis is

$$M_y = P_h l = P \sin \varphi l.$$

Here P_v and P_h are the vertical and horizontal components of the force P; l is the length of the beam.

The bending moment M_x produces tensile stresses in the upper half of the beam, including the point C, and compressive stresses in the lower half of the beam since the beam bends convex upward.

The magnitude of the stress at the point C is determined by the following formula, already familiar to us from the theory of plane bending:

$$\sigma_1 = \frac{M_x}{J_x} y,$$

where y is the distance from the neutral x-axis to the point C, J_x is the moment of inertia of the cross section of the beam with respect to the x-axis.

The bending moment M_y produces tensile stresses in the right-hand half of the beam, and therefore at the point C, and compressive stresses in the left-hand half of the beam since the beam bends convex to the right in the horizontal plane. This can readily be proved by bending a flexible ruler in the horizontal plane.

The magnitude of the stress at the point C due to the bending moment M_y is determined by a similar formula

$$\sigma_2 = \frac{M_y}{J_y} x,$$

where x is the distance from the y-axis (the neutral axis for bending in the horizontal plane) to the point C; J_y is the moment of inertia of the cross section of the beam with respect to the y-axis.

The total stress at the point C is, obviously, given by

$$\sigma = \frac{M_x}{J_x} y + \frac{M_y}{J_y} x. \tag{9.1}$$

These formulas are valid for any other shape of the cross section of a beam.

If a cross section has corner points for which $x_{\max}$ and $y_{\max}$ are reached simultaneously (rectangle, I-section), the maximum numerical values of stresses occur at these points:

$$\frac{\max}{\min} \sigma = \pm \frac{M_x}{W_x} \pm \frac{M_y}{W_y}, \tag{9.2}$$

where $W_x = \frac{2J_x}{h}$ is the section modulus with respect to the x-axis, $W_y = \frac{2J_y}{b}$ is the section modulus with respect to the y-axis. It is evident that those corner points of the section will be dangerous where stresses of the same sign are summed up.

For the case represented in Fig. 9.2 such points will be B and E, point B being in the stretched zone and point E in the compressed zone. Therefore the stresses at points B and E are

$$\sigma_B = \frac{M_x}{W_x} + \frac{M_y}{W_y},$$

$$\sigma_E = -\frac{M_x}{W_x} - \frac{M_y}{W_y}.$$

For a section of an arbitrary configuration having no projecting corner points it is necessary first to find "dangerous" points, i.e., those points of the cross section at which the numerically largest tensile and compressive stresses occur (Fig. 9.3).

This can be done as follows. First determine the position of the zero line in oblique bending, i.e., determine the locus of points of the cross section at which normal stresses are zero. In other words, determine the line dividing the stretched and compressed parts of the section. Let this be the line *n-n*.

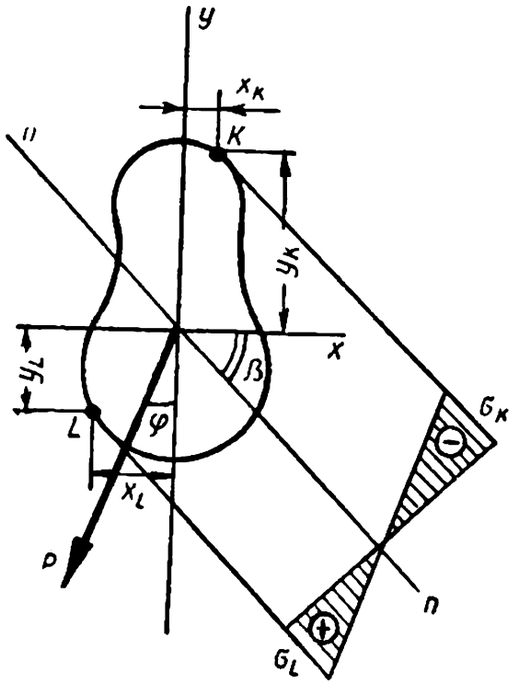


Fig. 9.3

Bearing in mind that in bending stresses increase with distance from the zero line, we conclude that points at which stresses should be checked are those most remote from the zero line, i.e., points K and L. If the material has the same strength in tension and compression the dangerous one of the above points is that at which the greater (in absolute value) stress occurs.

The equation of the zero line will be obtained by setting the right-hand member of formula (9.1) equal to zero:

$$\frac{M_x}{J_x} y_0 + \frac{M_y}{J_y} x_0 = 0$$

or

$$M_x \left(\frac{y_0}{J_x} + \frac{K x_0}{J_y} \right) = 0, \tag{9.3}$$

where

$$K = \frac{M_y}{M_x},$$

x_0 and y_0 are current co-ordinates of points of the zero line. Since $M_x \neq 0$ we have

$$\frac{y_0}{J_x} + \frac{K x_0}{J_y} = 0. \tag{9.4}$$

This is the equation of the zero line, the equation of a straight line passing through the origin. It can be represented in a slightly different form. By dividing both members of the equation by x_0 we obtain

$$\frac{y_0}{x_0 J_x} + \frac{K}{J_y} = 0. \tag{9.5}$$

But the ratio $\frac{y_0}{x_0} = \tan\beta$ is the tangent of the angle of inclination of the zero line to the x-axis, i.e., the slope of the straight line $\tan\beta = k$. Therefore

$$\frac{\tan\beta}{J_x} + \frac{K}{J_y} = 0$$

or

$$k = \tan\beta = -K\frac{J_x}{J_y}. \tag{9.6}$$

Thus, the equation of the zero line written in the canonical form known from analytic geometry (the equation of a straight line in the slope-intercept form) will be

$$y_0 = kx_0 = -K\frac{J_x}{J_y}x_0. \tag{9.7}$$

For the case presented in Fig. 9.3 we have $K = \tan\varphi$ and the expression for the slope of the zero line becomes

$$\tan\beta = -\tan\varphi\frac{J_x}{J_y}. \tag{9.8}$$

As is seen from this equation, at $J_x \neq J_y$ the angle β is not equal to the angle φ, i.e., the zero line is not perpendicular to the line of loading as was the case in plane (direct) bending. Only in a specific case when $J_x = J_y$ (circle, square, etc.) is the zero line perpendicular to the line of loading, but in such a case oblique bending is impossible as any centroidal line of the section is a principal axis of inertia.

It is advisable to bear in mind that the line of loading and the zero line always pass through different quadrants of the section (Fig. 9.3). In the present case the line of loading (i.e., the line of action of the force P) passes through the first and third quadrants while the zero line passes through the second and fourth quadrants.

After the position of the zero line has been determined and the points of the cross section most remote from the

zero line (dangerous points) found, it is possible to make a check on the strength of the section.

If the material of the beam has different resistance to tension and compression, the strength should be checked at the two points most remote from the zero line (points K and L in Fig. 9.3).

In the above case (see Fig. 9.3) the conditions of strength will be written in the form

$$\begin{aligned} |\sigma_K|^* &= -\frac{M_x}{J_x} y_K - \frac{M_y}{J_y} x_K \leqslant [\sigma_c], \\ \sigma_L &= \frac{M_x}{J_x} y_L + \frac{M_y}{J_y} x_L \leqslant [\sigma_t]. \end{aligned} \tag{9.9}$$

If $[\sigma_t]=[\sigma_c]$, i.e., for ductile materials, the analysis, as stated previously, is carried out for one point, for instance, in the case represented in Fig. 9.3 for point K, as $|\sigma_K| > |\sigma_L|$.

For rectangular sections, I-sections, etc., the condition of strength is obtained on the basis of formula (9.2):

$$\max \sigma = \frac{M_x}{W_x} + \frac{M_y}{W_y} \leqslant [\sigma]. \tag{9.10}$$

In case of a brittle material, such as cast iron, the allowable tensile stress $[\sigma_t]$ should be substituted in formula (9.10).

The required dimensions of the cross section are determined by the use (according to the shape of the section) of formulas (9.9) or (9.10) and the analysis is usually carried out by the method of trial and error.

Having assumed some dimensions of the section it is necessary to check whether the conditions of strength, formulas (9.9) or (9.10), are satisfied. If the difference between the design and allowable stresses is considerable, other dimensions of the section are assumed and the analysis is repeated. Such trials are carried on until the difference between σ_{max} and $[\sigma]$ is 5-10%. In the selection of rolled beam sections use is made of tables on profile shapes.

---

* $|\sigma_K|$ designates that the stress σ_K is taken in absolute value.

The selection of a circular cross section is made with the aid of the plane-bending formula with the total bending moment

$$M_b = \sqrt{M_x^2 + M_y^2}.$$

The problem of choosing a shape of a cross section for which the expenditure of material in oblique bending will be at a minimum is of certain practical importance. It can readily be shown (the student is recommended to do it) that for a rectangular cross section (Fig. 9.2) the minimum expenditure of material (the minimum cross-sectional area) will occur if

$$\frac{h}{b} = \frac{M_x}{M_y}. \tag{9.11}$$

However, a rectangular cross section is not the most efficient in oblique bending. The most economical sections are box thin-walled ones.

Deflections in oblique bending are determined analytically in each plane separately: by integrating the differential equation of the deflection curve, or by use of the universal formula, or by Mohr's method. The total deflection is determined as the geometric sum of the component deflections:

$$y_{\text{tot}} = \sqrt{y_h^2 + y_v^2}, \tag{9.12}$$

where y_h and y_v are the deflections in the horizontal and vertical planes respectively.

The condition of rigidity is

$$y_{\max} \leqslant [y], \tag{9.13}$$

where $[y]$ is the allowable deflection.

Example 9.1. Select a cross section for a beam subjected to bending in two planes (Fig. 9.4). The allowable stress is $[\sigma] = 1{,}600$ kg/cm^2.

Solution. By constructing the bending moment diagrams in the vertical and horizontal planes we obtain the maximum values of moments

$$M_x = 2 \text{ tm} \quad \text{and} \quad M_y = 1 \text{ tm}.$$

Let us compare several types of cross sections.

1. *Rectangular cross section.* Assume the optimum relation (9.11):

$$\frac{h}{b}=\frac{M_x}{M_y}=2, \quad \text{i. e.,} \quad h=2b.$$

Then according to formula (9.10) we have

$$\max\sigma=\frac{M_x}{W_x}+\frac{M_y}{W_y}=[\sigma];$$

$$\max\sigma=\frac{2\times10^5\times6}{b\times4b^2}+\frac{1\times10^5\times6}{2b\times b^2}=1{,}600 \text{ kg/cm}^2.$$

Hence $b=7.2$ cm; $h=14.4$ cm. The cross-sectional area is $F_1=104$ cm^2.

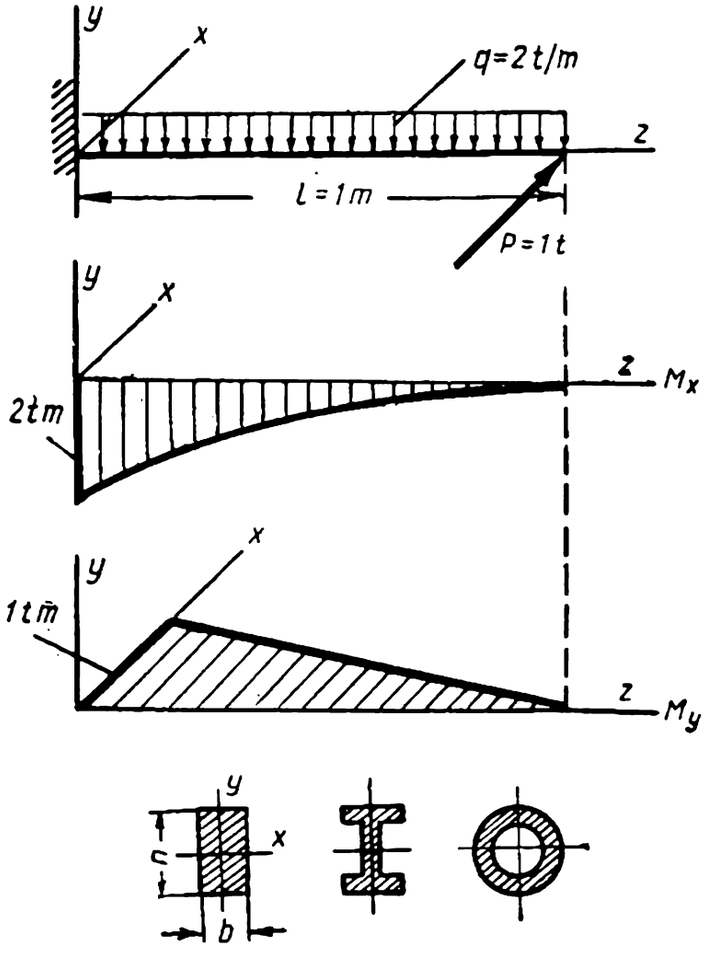


Fig. 9.4

2. *I-section.* Take I-section No. 36 for which $W_x=743$ cm^3; $W_y=71.1$ cm^3; $F=61.9$ cm^2. According to formula (9.10)

$$\max\sigma=\frac{2\times10^5}{743}+\frac{1\times10^5}{71.1}=270+1{,}400=1{,}670 \text{ kg/cm}^2$$

which exceeds the allowable stress by 4.4% (permissible).

3. *A circular section*, the ratio of the inside diameter to the outside diameter being equal to $c=\frac{d}{D}=0.9$.

A bar of circular (solid or ring) cross section must be designed with the use of the formulas of plane bending with the total moment:

$$M_b=\sqrt{M_x^2+M_y^2}=\sqrt{2^2+1^2}=2.23 \text{ tm}.$$

Determine the diameter of the section:

$$\max \sigma=\frac{2.23\times 10^5}{0.1D^3(1-0.9^4)}=1{,}600 \text{ kg/cm}^2,$$

whence $D=16$ cm; $d=0.9\times 16=14.4$ cm; $F=37.7$ cm^2.

If the area of the circular hollow section is taken as 100%, the area of the I-section will be 165% and the area of the rectangular section will be 276%.

It can be observed from this example what great possibilities exist for reducing the expenditure of material by using rational sections.

The student is recommended to continue the investigation of this problem, taking various box sections, the ratio between depth and width differing each time.

77. Bending with Tension (Compression)

Fig. 9.5 shows a case of combined bending and central tension. The transverse load producing bending may be of a more complicated type.

To determine the total stresses we shall use the principle of superposition. The tensile stresses due to force N, as is known, are equal all over the cross section and are given by

$$\sigma=\frac{N}{F}. \tag{2.2}$$

The stresses produced by the bending moment according to formula (6.8) are

$$\sigma=\frac{M_x}{J_x}y.$$

Consequently, the total stress at any point is

$$\sigma=\frac{N}{F}+\frac{M_x}{J_x}y. \tag{9.12}$$

In this case the dangerous section is at the built-in end where the bending moment has a maximum equal to $M_{max}=Pl$. The most stressed points in this section will be those lying on line AB, since at these points the tensile

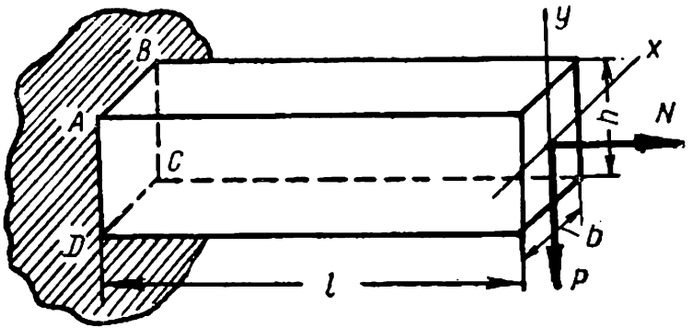


Fig. 9.5

stresses due to force N and the maximum tensile bending stresses will be summed up:

$$\sigma_{max}=\frac{N}{F}+\frac{M_x}{W_x}.$$

At points on line DC the stresses will be less and will be given by

$$\sigma=\frac{N}{F}-\frac{M_x}{W_x}.$$

The condition of strength is

$$\sigma_{max}=\frac{N}{F}+\frac{M_x}{W_x}\leqslant[\sigma_t]. \qquad (9.13)$$

If a transverse load is complicated, it is necessary first of all to construct the bending moment diagram to determine the dangerous section and the magnitude of the maximum bending moment.

The relations obtained are valid for a compressive load as well, only the stress $\frac{N}{F}$ will be negative and the maximum (numerically) stresses will occur at the points on the line DC. It should be pointed out that under the action of a compressive load the foregoing formulas are valid only for bars of great rigidity, so that the influence of the axial compressive force on the deflection is insignificant and can be neglected.

In case of combined tension (compression) and oblique bending the stresses are determined according to the formula

$$\sigma = \pm \frac{N}{F} \pm \frac{M_x}{J_x} y \pm \frac{M_y}{J_y} x. \qquad (9.14)$$

For bars made of ductile materials (i.e., having the same strength in tension and compression) with cross sections having corner points which are simultaneously the most remote points from the principal x and y axes (such as a rectangle, I-section, etc.) we obtain the following condition of strength:

$$\max \sigma = \frac{N}{F} + \frac{M_x}{W_x} + \frac{M_y}{W_y} \leqslant [\sigma_t].$$

78. Eccentric Compression (Tension)

Fig. 9.6*a* shows a case of eccentric tension.

A force parallel to the axis of the bar is applied at point $A(x_N, y_N)$, which does not lie on one of the principal centroidal axes of inertia of the cross section.

By transmitting the force to the centroid of the cross section this case is reduced to combined axial tension and bending in two principal planes. The bending moment with respect to the x-axis is

$$M_x = N y_N;$$

the bending moment with respect to the y-axis is

$$M_y = N x_N.$$

The distance x_N or y_N is called the eccentricity of the force with respect to the y-axis or x-axis, as the case may be. In the general case of eccentric tension or compression the stress developed at any point of a cross section whose co-ordinates are x and y is determined by a formula analogous to formula (9.14):

$$\sigma = \pm \frac{N}{F} \pm \frac{N y_N}{J_x} y \pm \frac{N x_N}{J_y} x. \qquad (9.15)$$

If the cross section has corner points (rectangle, I-section, etc.), then at one of these points the maximum tensile stress is produced, and at the other the maximum compressive stress (points D and E in Fig. 9.6*b*).

By combining the formulas for the maximum tensile and compressive stresses into one we can write

$$\frac{\max}{\min}\sigma = \pm\frac{N}{F} \pm \frac{M_x}{W_x} \pm \frac{M_y}{W_y}. \qquad (9.16)$$

At the section shown in Fig. 9.6*b* the maximum stresses will occur at point E since here the tensile stresses due to force N are summed up with the tensile stresses due to the bending moments M_x and M_y:

$$\sigma_E = \frac{N}{F} + \frac{M_x}{W_x} + \frac{M_y}{W_y}.$$

The minimum (in the algebraic sense) stresses will occur at point D:

$$\sigma_D = \frac{N}{F} - \frac{M_x}{W_x} - \frac{M_y}{W_y}.$$

They may be both tensile and compressive (depending on the values of x_N, y_N). The condition of strength will be

$$\max\sigma = \frac{N}{F} + \frac{M_x}{W_x} + \frac{M_y}{W_y} \leqslant [\sigma]. \qquad (9.17)$$

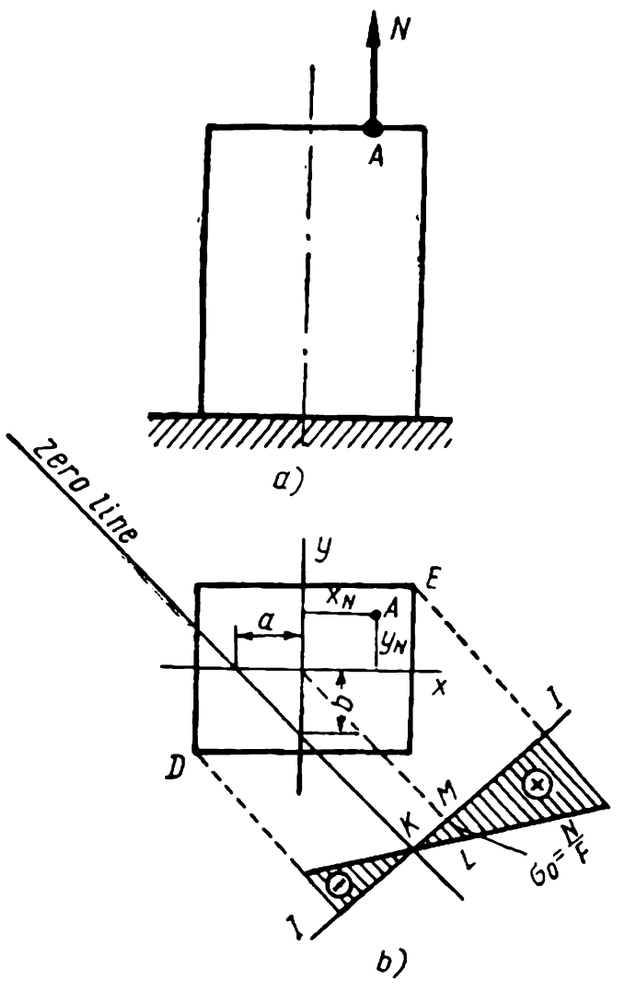


Fig. 9.6

If the point of application of the force is on one of the principal axes of inertia of the cross section, for instance on the y-axis, the preceding formula is simplified:

$$\max\sigma = \frac{N}{F} + \frac{M_x}{W_x} \leqslant [\sigma]. \qquad (9.18)$$

For an arbitrary shape of a cross section it is necessary to locate the zero line to determine the position of the dangerous points. The equation of the zero line is obtained by equating the right side of formula (9.15) to zero:

$$\frac{N}{F} + \frac{Ny_N}{J_x} y_0 + \frac{Nx_N}{J_y} x_0 = 0. \qquad (9.19)$$

Here x_0 and y_0 are the current co-ordinates of the points on the zero line. Introduce notations

$$i_x^2 = \frac{J_x}{F},$$

$$i_y^2 = \frac{J_y}{F}.$$

The quantities

$$i_x = \sqrt{\frac{J_x}{F}},$$

$$i_y = \sqrt{\frac{J_y}{F}}$$

characterise the geometry of the cross section and are called *the radii of gyration* of the cross section with respect to the x and y axes. The dimension of the radius of gyration is the unit of length, usually cm. Formula (9.19) can now be written in the form

$$\frac{N}{F}\left(1 + \frac{x_N x_0}{i_y^2} + \frac{y_N y_0}{i_x^2}\right) = 0.$$

Since $\frac{N}{F} \neq 0$

$$\frac{x_N x_0}{i_y^2} + \frac{y_N y_0}{i_x^2} + 1 = 0. \tag{9.20}$$

This is the equation of the zero line. It can be presented as the equation of a straight line in the intercept form:

$$\frac{x_0}{a} + \frac{y_0}{b} = 1, \tag{9.21}$$

where

$$a = -\frac{i_y^2}{x_N}, \tag{9.22a}$$

$$b = -\frac{i_x^2}{y_N} \tag{9.22b}$$

are the x and y intercepts of the zero line, respectively.

Since the radius of gyration is always positive, the quantities a and x_N, as well as b and y_N, have opposite signs.

After the zero line has been located, it is easy to construct the normal stress diagram. For that draw a line *1-1* (the axis of the diagram) perpendicular to the zero line

(see Fig. 9.6*b*). Transfer the extreme points of the cross section *D* and *E* to this line by dropping perpendiculars from these points on the line *1-1*.

To construct the stress diagram we have two points: point *K* on the zero line where $\sigma=0$ and point *L* resulting from the condition that the segment *LM* must, to a certain scale, give the magnitude of the stress developed at the centroid of the cross section. Since $x=y=0$ at the centroid of the cross section, we obtain this stress from formula (9.15) as

$$\sigma_0=\frac{N}{F}.$$

Example 9.2. Determine the position of the zero line for the section shown in Fig. 9.7, if $x_N=3$ cm, $y_N=-2$ cm.

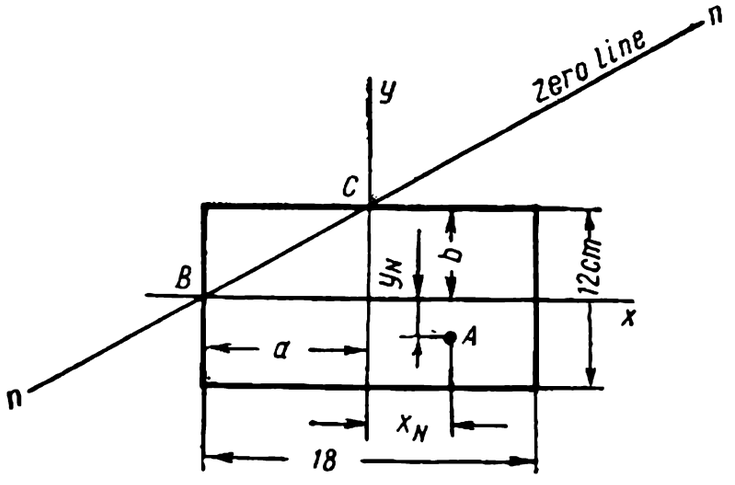


Fig. 9.7

Solution. According to formulas (9.22) we determine the intercepts of the zero line on the principal axes of inertia *x* and *y*:

$$a=-\frac{i_y^2}{x_N}=-\frac{\frac{12\times 18^3}{12\times 12\times 18}}{3}=-9\text{ cm},$$

$$b=-\frac{i_x^2}{y_N}=-\frac{\frac{18\times 12^3}{12\times 12\times 18}}{-2}=+6\text{ cm}.$$

Lay off these segments and draw the zero line *n-n* through points *B* and *C*.

Example 9.3. In drilling a part the spindle *A* of the drilling machine is subjected to an axial load of 1,000 kg

(Fig. 9.8). Determine the diameter of the solid cast-iron column B. The allowable tensile stress is $[\sigma_t]=400$ kg/cm^2.

Solution. The column is subjected to eccentric tension. According to formula (9.18) we have

$$\max\sigma=\frac{N}{F}+\frac{M_b}{W_x}\leqslant[\sigma_t].$$

Substituting the numerical values we obtain

$$\max\sigma=\frac{1{,}000\times4}{3.14\times d^2}+\frac{1{,}000\times40}{0.1\times d^3}\leqslant400.$$

By the trial and error method we determine $d=10.1$ cm.

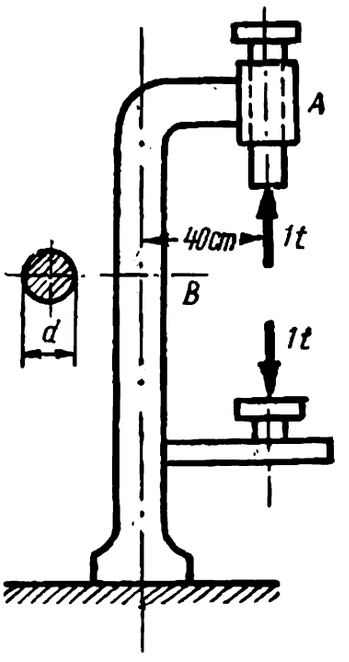


Fig. 9.8

Example 9.4. A bar of rectangular cross section 10×20 cm, clamped at one end, is stretched by force $P_1=5$ t, applied with the eccentricity 2 cm with respect to the x-axis and bent by force $P_2=400$ kg (Fig. 9.9). Determine the normal stresses at points A, B, C and D of the section at the built-in end.

Hint. Force P_1 produces tension and bending in the vertical plane (about the x-axis); force P_2 produces bend-

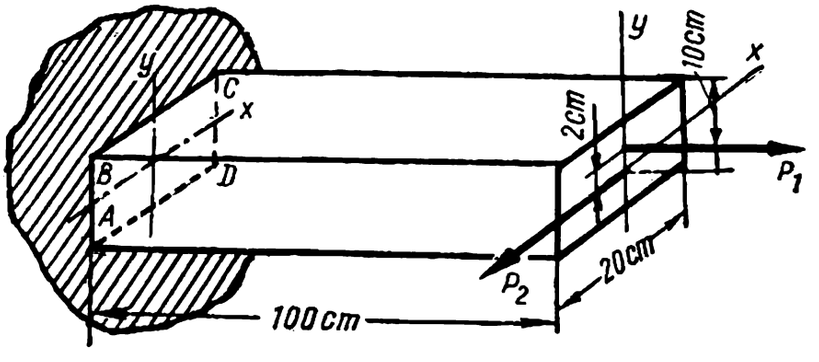


Fig. 9.9

ing of the bar in the horizontal plane (about the y-axis).

Answer: $\sigma_A=-65$ kg/cm^2 (compression), $\sigma_B=-5$ kg/cm^2 (compression), $\sigma_C=+115$ kg/cm^2 (tension), $\sigma_D=+55$ kg/cm^2 (tension).

79. Core of a Cross Section

If the zero line does not intersect the cross section, the stresses will have the same sign all over this section.

Let us determine the co-ordinates of a point of application of a force with which the stresses developed on the cross section are of the same sign and the zero line is a tangent line to the contour of the cross section. From formulas (9.22) the following expressions can readily be obtained for the co-ordinates x_N and y_N of the point of application of a force:

$$x_N = -\frac{i_y^2}{a}, \tag{9.23a}$$

$$y_N = -\frac{i_x^2}{b}. \tag{9.23b}$$

Here a and b are the intercepts of the tangent line to the contour of the cross section on the co-ordinate axes. For different locations of the tangent to a continuous contour of a cross section these formulas give the co-ordinates of an infinite number of points which combined represent a certain contour round the centroid of the cross section.

The portion of the cross-sectional area lying within this contour is called the core of a cross section.

The core of a cross section has the following property: if a force parallel to the axis of a bar is applied within the core, the normal stresses at all points of the cross section will be of the same sign.

For a polygonal cross section with n sides formulas (9.23) give the co-ordinates of n points of the core of the cross section.

To obtain the contour of the core of the cross section the points thus found should be joined by straight lines. This follows from the fact that when passing from one side of the contour to another the zero line rotates about a fixed point, the relation between the co-ordinates x_N and y_N being linear (see equation 9.19).

Brittle materials, as is known, are weak in tension. Therefore, in bars of such materials it is important to apply a compressive force within the core of the cross section in order to avoid tensile stresses.

Example 9.5. Determine the dimensions of the core of a circular cross section d in diameter (Fig. 9.10).

Solution. Since the circle has central symmetry, it is sufficient to determine the location of one point of the core of the cross section. The tangent line AB intersects the x and y axes at distances:

$$a = -\frac{d}{2}, \quad b = \infty.$$

According to formulas (9.23) we obtain

$$x_N = -\frac{i_y^2}{a} = -\frac{\frac{J_y}{F}}{-\frac{d}{2}} = \frac{\pi d^4 \times 4 \times 2}{64\pi d^2 d} = \frac{d}{8},$$

$$y_N = 0.$$

Consequently, the core of the cross section will be a circle of radius $\frac{d}{8}$.

Example 9.6. Determine the dimensions of the core of a rectangular cross section (Fig. 9.11).

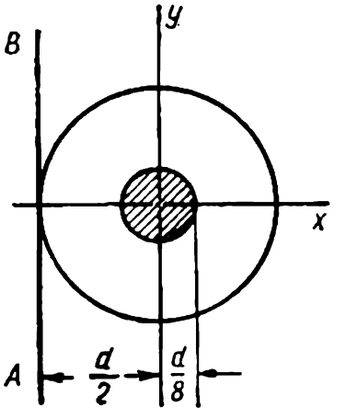


Fig. 9.10

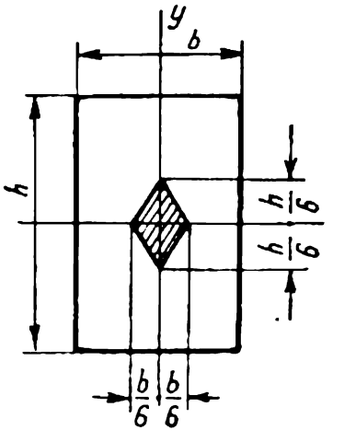


Fig. 9.11

Answer. The core of the cross section is a rhombus with the diagonals $\frac{h}{3}$ and $\frac{b}{3}$.

80. Torsion and Shear. Design of Closely Coiled Springs

The combined action of a shearing force and a twisting moment is found in the cross sections of the coils of a spring with a slight pitch angle, subjected to a compressive or tensile load P (Fig. 9.12).

To calculate the internal forces acting on the cross sections of the rod of which the spring is made we shall apply the method of sections. Make a section and consider the equilibrium of the lower portion of the spring (Fig. 9.13).

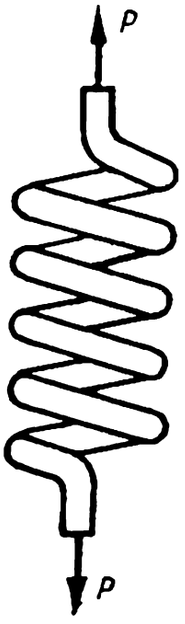


Fig. 9.12

Let D be the mean diameter of the spring, n—the number of its coils, d—the diameter of the cross section of the rod. The angle of pitch is disregarded.

To hold this portion of the spring in equilibrium it is necessary to apply at the cross section of the rod a shearing force Q equal in magnitude to force P and a twisting moment M_t equal to the moment produced by force P with respect to the centroid of the cross section (with respect to the longitudinal axis z of the coil).

All other internal forces will be equal to zero, which can be deduced from the equations of equilibrium derived on the assumption that the axis of the coil lies in a plane perpendicular to the axis of the spring.

The magnitude of force Q and moment M_t will be found from the equations of equilibrium. Equating to zero the

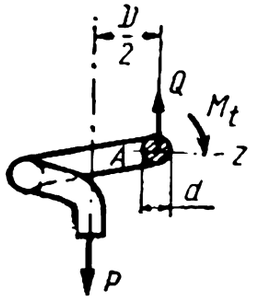


Fig. 9.13

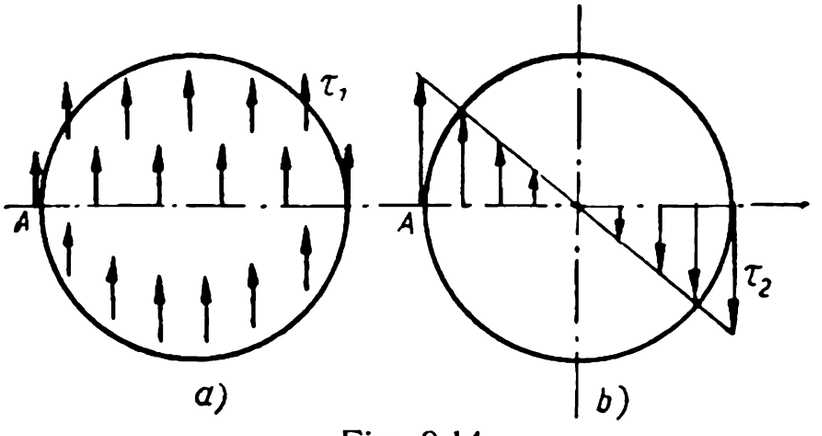


Fig. 9.14

sum of the projections on the vertical axis of the forces acting on the cut portion we obtain

$$Q = P.$$

Equating to zero the sum of the moments acting on the cut portion we obtain

$$M_t = P\frac{D}{2}.$$

Assume that the shearing stresses due to shear (caused by the shearing force) are uniformly distributed over the cross section (Fig. 9.14*a*):

$$\tau_1 = \frac{Q}{F} = \frac{4P}{\pi d^2}. \tag{9.24}$$

Assume also that the shearing stresses due to torsion (caused by the twisting moment) are distributed over the cross section of the coil in the same manner as in torsion of a straight bar of circular cross section, i.e., they increase from the centre to the edge of the section according to the linear law (Fig. 9.14*b*). Therefore, the maximum stresses due to torsion are given by

$$\tau_2 = \frac{M_t}{W_p} = \frac{P\frac{D}{2}}{\frac{\pi d^3}{16}} = \frac{8PD}{\pi d^3}. \tag{9.25}$$

The dangerous point will be the point A on the contour where the directions of τ_1 and τ_2 coincide. Hence, the maximum shearing stresses will be

$$\tau_{max} = \frac{4P}{\pi d^2} + \frac{8PD}{\pi d^3} = \frac{8PD}{\pi d^3}\left(1 + \frac{d}{2D}\right). \tag{9.26}$$

In most cases the second term in brackets is considerably less than unity and it can be disregarded, which is equivalent to disregarding the effect of shear in comparison with the effect of torsion.

Then we can take as an approximation

$$\tau_{max} \approx \tau_2 \approx \frac{8PD}{\pi d^3}.$$

In practice springs with a slight angle of pitch are designed according to the formula

$$\tau_{max} = k\frac{8PD}{\pi d^3} \leqslant [\tau], \tag{9.27}$$

where k is a correction factor accounting for the effect of force Q and the effect of the curvature of the coil on the distribution and magnitude of the shearing stresses caused by the twisting moment.

The values of k are given in Table 9.1.

Table 9.1

Correction Factor k for Strength Design of Springs

$\frac{D}{d}$	3	4	5	6	7	8	9	10
k	1.58	1.40	1.31	1.25	1.21	1.18	1.16	1.14

To determine deflection δ (change in length) of the spring we equate the work done by the external force and the potential energy of strain in torsion.

The work done by force P over displacement δ is given, as is known, by

$$A = \frac{P\delta}{2}.$$

The potential energy of strain in torsion is (Sec. 39)

$$\Pi = \frac{1}{2}\frac{M_t^2 l}{GJ_p} = \frac{1}{2}\frac{\left(P\frac{D}{2}\right)^2 l}{GJ_p}.$$

Here $l = \pi Dn$ is the length of the rod of which the spring is made, n is the number of coils.

Since $A = \Pi$, we have

$$\frac{P\delta}{2} = \frac{P^2D^2\pi Dn}{8G\frac{\pi d^4}{32}}, \quad \text{from which} \quad \delta = \frac{8PD^3n}{Gd^4}. \qquad (9.28)$$

Example 9.7. Check the strength of a spring and determine its deflection if $P = 300$ kg, $D = 200$ mm, $[\tau] = 2{,}500$ kg/cm^2, $d = 20$ mm, the number of coils $n = 8$, $G = 8 \times 10^5$ kg/cm^2.

Solution. The stress is

$$\tau_{max} = 1.14\,\frac{8 \times 300 \times 20}{3.14 \times 2^3} = 2{,}180 \text{ kg/cm}^2,$$

which is 8.8 per cent lower than the allowable stress. Here k is taken to be 1.14 according to Table 9.1 for $\frac{D}{d} = 10$. The deflection of the spring is determined by the use of formula (9.28):

$$\delta = \frac{8 \times 300 \times 20^3 \times 8}{8 \times 10^5 \times 2^4} = 12 \text{ cm}.$$

Example 9.8. A spring is made of steel wire 4 mm in diameter (d). The mean diameter of the spring is $D=45$ mm. The number of active coils * of the spring is $n=6$. In the unloaded condition the clearance between the coils is $S_0=1.0$ mm. What compressive force should be applied to the spring to make the clearance disappear?

Hint. The deflection of the spring when the coils come into contact is

$$\delta=(n-1)S.$$

Answer: $P=2.35$ kg.

81. Torsion with Bending

The bar represented in Fig. 9.15 acts in torsion and bending. In machine construction parts subjected to combined torsional and bending loads are encountered very often. A typical example is the shafts of various machines.

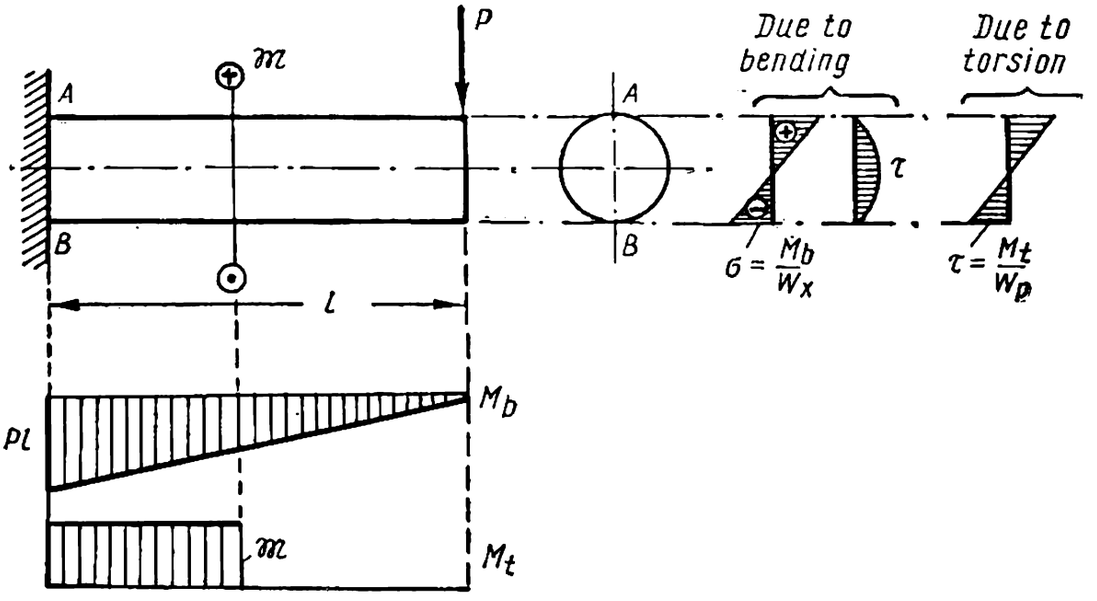


Fig. 9.15

Let us start by determining separately, using the principle of superposition, the stresses developed in the bar due to torsion and due to bending. In bending at cross sections of a beam there occur, as is known, normal stresses

* The total number of coils of a compression spring exceeds the number of active coils by 1.5-2, for the extreme coils contacting with the supporting plates do not contribute to the deformation of the spring.

reaching the maximum value in the extreme fibres of the beam:

$$\sigma = \frac{M_b}{W_x}$$

and shearing stresses reaching the maximum value at the neutral axis, which are determined by Zhuravsky's formula. For circular and massive cross sections, in general, their magnitude is insignificant as compared to shearing stresses due to torsion. In torsion at cross sections there occur shearing stresses reaching the maximum value at points on the contour of the cross section:

$$\tau = \frac{M_t}{W_p} = \frac{M_t}{2W_x}.$$

In the case illustrated in Fig. 9.15 the section where the maximum bending moment occurs coincides with the section where the maximum twisting moment occurs. This is the section at the built-in end. In this (dangerous)

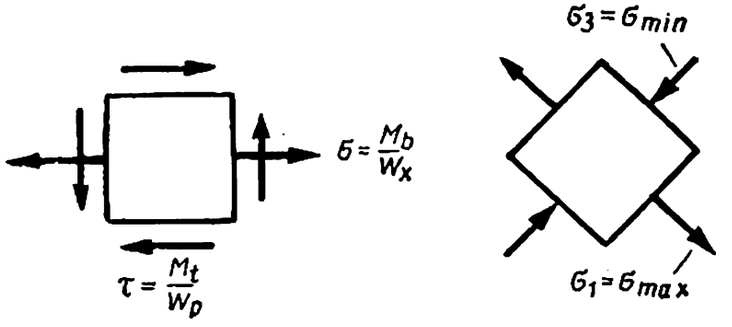


Fig. 9.16

section the dangerous points are at A and B. Consider the state of stress at point A (Fig. 9.16). The area of the transverse section passing through this point is subjected to the maximum shearing stresses due to torsion $\tau = \frac{M_t}{W_p}$ and the maximum (in this case tensile) normal stresses due to bending $\sigma = \frac{M_b}{W_x}$. On the area of the longitudinal section no normal stresses exist, whereas the shearing stresses (by virtue of the law of conjugate shearing stresses) have the same magnitude as in the transverse section.

Since the state of stress is biaxial, to check the strength we apply one of the strength theories. For shafts made of steel the third or fourth theory of strength can be applied.

For that it is necessary to determine the principal stresses for the given state of stress (see Fig. 9.16).

The principal stresses are determined according to the known formula

$$\sigma_{1,3} = \sigma_{\substack{\max \\ \min}} = \frac{\sigma}{2} \pm \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2}.$$

The problem of determining the angle of inclination of the principal areas will not be discussed here; it can be found by means of formula (2.35).

The condition of strength, based on the third theory (the maximum shearing stress theory), is

$$\sigma_{eq} = \sigma_1 - \sigma_3 \leqslant [\sigma].$$

Substituting here the values of σ_1 and σ_3 we obtain

$$\sigma_{eq} = \sqrt{\sigma^2 + 4\tau^2} \leqslant [\sigma]. \tag{9.29}$$

Taking into consideration that

$$\sigma = \frac{M_b}{W_x} \quad \text{and} \quad \tau = \frac{M_t}{2W_x}$$

we have

$$\sigma_{eq} = \frac{\sqrt{M_b^2 + M_t^2}}{W_x} \leqslant [\sigma]. \tag{9.30}$$

For the selection (design) of a cross section we obtain the following relation

$$W_x = \frac{\sqrt{M_b^2 + M_t^2}}{[\sigma]}. \tag{9.31}$$

It will be recalled that in the case of a shaft being subjected to bending in two mutually perpendicular planes

$$M_b = \sqrt{M_x^2 + M_y^2}.$$

According to the fourth theory of strength (the maximum distortion energy theory) the condition of strength for the case of plane stress is

$$\sigma_{eq} = \sqrt{\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3} \leqslant [\sigma].$$

Substituting the values of σ_1 and σ_3 expressed in terms of σ and τ in the cross section of the shaft we obtain

$$\sigma_{eq} = \sqrt{\sigma^2 + 3\tau^2} \leqslant [\sigma]. \tag{9.32}$$

But $\sigma = \frac{M_b}{W_x}$ and $\tau = \frac{M_t}{W_p}$, therefore

$$\sigma_{eq} = \frac{\sqrt{M_b^2 + 0.75M_t^2}}{W_x} \leqslant [\sigma]. \tag{9.33}$$

For the selection of a cross section we have

$$W_x = \frac{\sqrt{M_b^2 + 0.75M_t^2}}{[\sigma]}. \tag{9.34}$$

For materials having different strength in tension and compression (some grades of alloy steel, cast iron, some alloys) use should be made of Mohr's theory of strength or N. N. Davidenkov's combined theory of strength. According to Mohr's theory of strength

$$\sigma_{eq} = \sigma_1 - \nu\sigma_3 \leqslant [\sigma_t],$$

where

$$\nu = \frac{[\sigma_t]}{[\sigma_c]}.$$

Substituting here the values of σ_1 and σ_3 we obtain

$$\sigma_{eq} = \frac{1-\nu}{2}\sigma + \frac{1+\nu}{2}\sqrt{\sigma^2 + 4\tau^2} \tag{9.35}$$

or

$$\sigma_{eq} = \frac{1-\nu}{2}\frac{M_b}{W_x} + \frac{1+\nu}{2W_x}\sqrt{M_b^2 + M_t^2} \leqslant [\sigma]. \tag{9.36}$$

For the selection of cross sections we obtain the following relation

$$W_x = \frac{M_b(1-\nu) + (1+\nu)\sqrt{M_b^2 + M_t^2}}{2[\sigma]}. \tag{9.37}$$

82. Torsion with Tension (Compression)

In this case there are two internal forces acting simultaneously at cross sections of a bar: a twisting moment and a longitudinal force (tensile or compressive).

For a bar of circular cross section the maximum shearing stresses occur at the points of the contour of the cross section and are given by

$$\tau = \frac{M_t}{W_p}.$$

The longitudinal force produces at all points of the cross section the normal stress

$$\sigma = \frac{N}{F}.$$

As in the case of torsion with bending, it is necessary now to determine the principal stresses and to apply a suitable theory of strength.

As a result we obtain for the reduced stresses formula (9.29) (according to the third theory of strength) or formula (9.32) (according to the fourth theory).

The values of σ and τ derived above should be substituted into these formulas.

Finally we obtain the condition of strength for torsion with tension (compression):

(a) according to the third theory of strength

$$\sqrt{\left(\frac{N}{F}\right)^2 + 4\left(\frac{M_t}{W_p}\right)^2} \leqslant [\sigma]; \tag{9.38}$$

(b) according to the fourth theory of strength

$$\sqrt{\left(\frac{N}{F}\right)^2 + 3\left(\frac{M_t}{W_p}\right)^2} \leqslant [\sigma]. \tag{9.39}$$

It is advisable to give preference to formula (9.39) as the fourth theory is in good agreement with experimental data for ductile materials and leads to more economical solutions.

For materials having different strength in tension and compression, we obtain according to Mohr's theory on the basis of formula (9.35)

$$\frac{1-\nu}{2}\frac{N}{F} + \frac{1-\nu}{2}\sqrt{\left(\frac{N}{F}\right)^2 + 4\left(\frac{M_t}{W_p}\right)^2} \leqslant [\sigma]. \tag{9.40}$$

For the case of combined bending, torsion and tension (compression) we obtain in a similar way the following computing formula (according to the fourth theory of strength)

$$\sqrt{\left(\frac{N}{F} + \frac{M_b}{W_x}\right)^2 + 3\left(\frac{M_t}{W_p}\right)^2} \leqslant [\sigma]. \tag{9.41}$$

In the selection of cross sections use should be made of the method of successive approximations, i.e., one must assign some value of the diameter and check whether the conditions of strength (9.39), (9.40) or (9.41) are satisfied. If the difference between the left-hand and right-hand members is found to be great, a new value of the diameter should be assigned, and so on. When a slide rule is used this procedure does not take much time.

In checking analysis when the diameter of a shaft is known, the factor of safety is calculated by the formula

$$n = \frac{\sigma_y}{\sigma_{eq}}, \tag{9.42}$$

where σ_y is the yield point, σ_{eq} is the equivalent (reduced) stress based on the corresponding theory of strength.

For ductile materials the equivalent stress can be calculated according to the third theory of strength with the use of formula (9.29).

Therefore, the factor of safety will be

$$n = \frac{\sigma_y}{\sqrt{\sigma^2 + 4\tau^2}}.$$

This formula can also be written as

$$n = \frac{1}{\sqrt{\frac{\sigma^2}{\sigma_y^2} + \frac{\tau^2}{\left(\frac{\sigma_y}{2}\right)^2}}} = \frac{1}{\sqrt{\frac{\sigma^2}{\sigma_y^2} + \frac{\tau^2}{\tau_y^2}}}, \tag{9.43}$$

here $\tau_y = \frac{\sigma_y}{2}$ (in accordance with the third theory of strength).

Formula (9.43) after some transformations reduces to

$$n = \frac{1}{\sqrt{\frac{1}{n_\sigma^2} + \frac{1}{n_\tau^2}}} = \frac{n_\sigma n_\tau}{\sqrt{n_\sigma^2 + n_\tau^2}}, \tag{9.44}$$

here $n_\sigma = \frac{\sigma_y}{\sigma}$ is the factor of safety based on normal stresses, $n_\tau = \frac{\tau_y}{\tau}$ is the factor of safety based on shearing stresses.

It should be noted that formulas (9.43) and (9.44) hold good when using the fourth theory of strength, the only difference being that in this case $\tau_y \approx 0.58\ \sigma_y$.

83. An Example of Design of a Shaft Subjected to Bending with Torsion

Shafts subjected to torsion and bending are designed to withstand maximum short-term forces (static strength) as well as alternating forces acting over a long period of time (endurance) (see Chapter XII).

Let us illustrate the static strength design of a shaft (Fig. 9.17*a*) by an example. The shaft is acted upon by

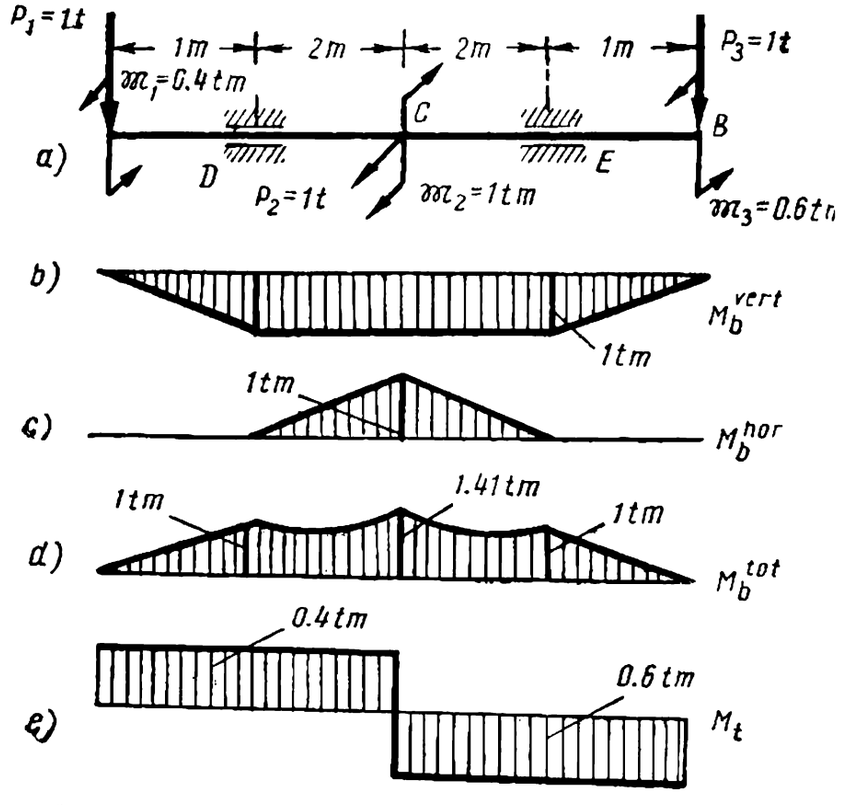


Fig. 9.17

two vertical forces P_1 and P_3, one horizontal force P_2 and three external moments: $\mathfrak{M}_1=0.4$ tm, $\mathfrak{M}_2=1$ tm, $\mathfrak{M}_3=0.6$ tm producing torsion. The material of the shaft is steel 45, the yield point $\sigma_y=3{,}600$ kg/cm^2, the ultimate strength $\sigma_u=6{,}100$ kg/cm^2.

According to data given in Sec. 12, we choose the total factor of safety n as the product of three partial factors

$$n=n_1n_2n_3.$$

Assume $n_1=1.3$ (allowance for a sufficiently high degree of accuracy in determining stresses), $n_2=1.4$ (for the ratio

$\frac{\sigma_y}{\sigma_u} = 0.6\Big)$, $n_3 = 1.3$ (allowance for a moderate degree of responsibility of the member).

Thus, the total factor of safety is

$$n = 1.3 \times 1.4 \times 1.3 = 2.36.$$

The allowable stress is

$$[\sigma] = \frac{3{,}600}{2.36} \approx 1{,}500 \text{ kg/cm}^2.$$

1. Construct the bending moment diagram due to the vertical forces M_b^{vert} (Fig. 9.17*b*).

2. Construct the bending moment diagram due to the horizontal forces. The M_b^{hor} diagram, conventionally represented as being in the plane of the drawing, is shown in Fig. 9.17*c*.

Since the cross section of the shaft is circular and plane (direct) bending takes place, the calculations should take into account the resultant (total) moment.

3. For clearness construct the total bending moment diagram as the geometric sum of the two preceding diagrams. The total bending moments are

$$M_D^{tot} = 1 \text{ tm}; \quad M_C^{tot} = \sqrt{1^2 + 1^2} = 1.41 \text{ tm}; \quad M_E^{tot} = 1 \text{ tm}.$$

The M_b^{tot} diagram is presented in Fig. 9.17*d*. The ordinates of this diagram are drawn conventionally in the plane of the drawing.

4. Construct the twisting moment diagram. The M_t diagram is shown in Fig. 9.17*e*.

Determine the diameter of the shaft at the dangerous section *C*, where the maximum bending moment $M_b^{tot} = 1.41$ tm and twisting moment $M_t = 0.6$ tm occur. For selecting the cross section we apply the fourth theory of strength

$$W_x = \frac{\sqrt{M_b^2 + 0.75 M_t^2}}{[\sigma]} = \frac{\sqrt{141.000^2 + 0{,}75 \times 60.000^2}}{1{,}500} = 100 \text{ cm}^3,$$

whence

$$d = \sqrt[3]{\frac{W_x}{0.1}} = \sqrt[3]{\frac{100}{0.1}} = 10 \text{ cm}.$$

84. Analysis of Thin-Walled Vessels

Vessels whose walls are subjected to pressure from fluids, dry materials and gases are frequently encountered in engineering practice (steam boilers, reservoirs, working chambers of engines, tanks, etc.). If vessels have the form

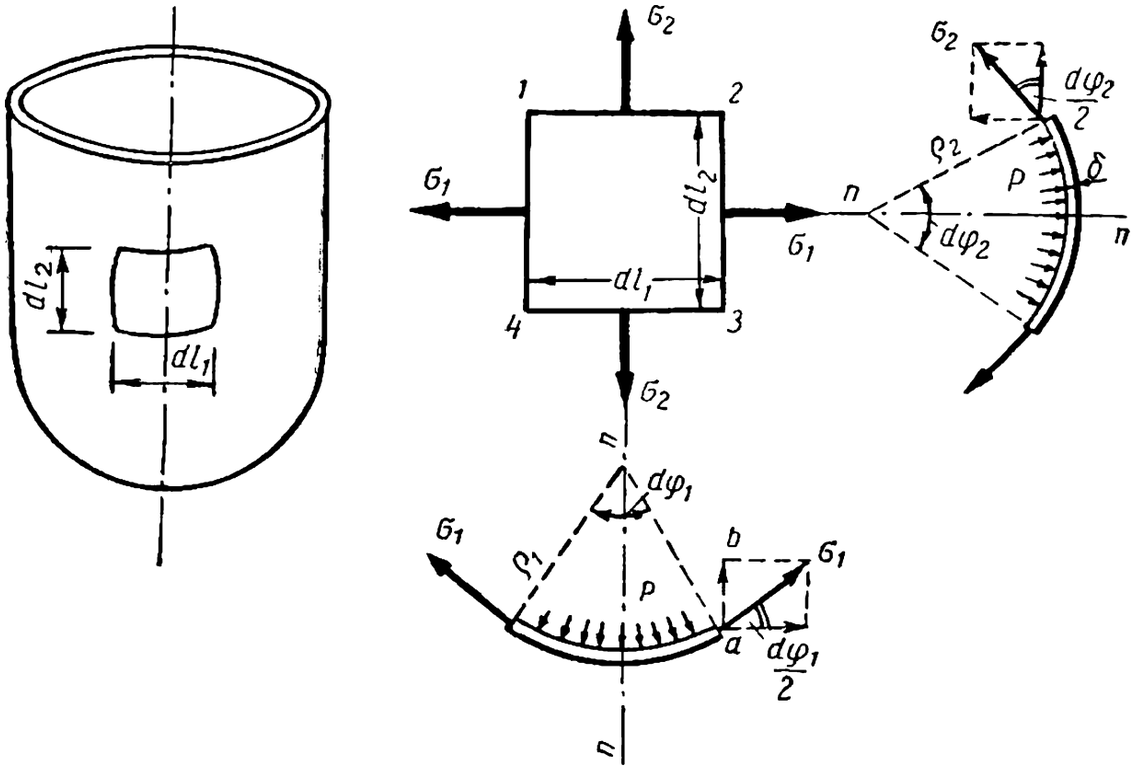


Fig. 9.18

of bodies of revolution and thin walls, the stresses developed in their walls under loading can be determined quite simply.

When the wall thickness is small it may be assumed without serious errors being introduced that only normal stresses (tensile or compressive) occur in the walls and that these stresses are distributed uniformly throughout the thickness of the wall.

Calculations based on these assumptions are in good agreement with experiments where the wall thickness does not exceed about 0.1 of the minimum radius of curvature of the wall.

Isolate an element with dimensions dl_1 and dl_2 in the wall of a vessel (Fig. 9.18). Let the wall thickness be denoted by δ, the radii of curvature of the surface of the element by ρ_1 and ρ_2. The element is subjected to an internal pressure of intensity p normal to its surface.

Replace the interaction of the element with the remaining part of the vessel by internal forces whose intensity is σ_1 and σ_2. In view of the small thickness of the walls, as has been stated, these stresses can be considered to be uniformly distributed throughout the wall thickness.

Derive the condition of equilibrium of the element by projecting the forces acting upon the element on the direction of the normal n-n to the surface of the element.

The projection of the load will be pdl_1dl_2.

The projection of the stress σ_1 on the direction of the normal will be represented by segment ab equal to $\sigma_1 \sin \frac{d\varphi_1}{2}$.

The projection of the force acting over the face *1-4* (and equally *2-3*) is

$$2\sigma_1 \delta\, dl_2 \sin \frac{d\varphi_1}{2}.$$

Likewise, the projection of the force acting over the face *1-2* (and equally *4-3*) is

$$2\sigma_2 \delta\, dl_1 \sin \frac{d\varphi_2}{2}.$$

By projecting all the forces applied to the isolated element on the direction of the normal n-n we obtain

$$p\,dl_1\,dl_2 - 2\sigma_1 \delta\, dl_2 \sin \frac{d\varphi_1}{2} - 2\sigma_2 \delta\, dl_1 \sin \frac{d\varphi_2}{2} = 0.$$

Since the dimensions of the element are small it may be assumed that

$$\sin \frac{d\varphi_1}{2} \approx \frac{d\varphi_1}{2} \quad \text{and} \quad \sin \frac{d\varphi_2}{2} \approx \frac{d\varphi_2}{2}.$$

Taking this into consideration we obtain from the equation of equilibrium

$$p\,dl_1\,dl_2 - \sigma_1 \delta\, dl_2\, d\varphi_1 - \sigma_2 \delta\, dl_1\, d\varphi_2 = 0.$$

Bearing in mind that $d\varphi_1 = \frac{dl_1}{\rho_1}$ and $d\varphi_2 = \frac{dl_2}{\rho_2}$ we have

$$p\,dl_1\,dl_2 - \frac{\sigma_1 dl_1\, dl_2 \delta}{\rho_1} - \frac{\sigma_2 dl_1\, dl_2 \delta}{\rho_2} = 0.$$

Cancelling out dl_1dl_2 and dividing by δ we obtain

$$\frac{\sigma_1}{\rho_1} + \frac{\sigma_2}{\rho_2} = \frac{p}{\delta}. \tag{9.45}$$

This formula is called Laplace's equation. Consider the calculation of vessels of two types which are often encountered in practice: spherical and cylindrical. * We shall limit ourselves to a case of internal gas pressure.

1. *Spherical vessel.*

In this case:

$$\rho_1 = \rho_2 = r \quad \text{and} \quad \sigma_1 = \sigma_2 = \sigma.$$

Formula (9.45) yields

$$\frac{2\sigma}{r} = \frac{p}{\delta},$$

whence

$$\sigma = \frac{pr}{2\delta}. \tag{9.46}$$

Since in this case the state of plane stress occurs, it is necessary to apply one of the strength theories for the strength analysis. The principal stresses have the following values: $\sigma_1 = \sigma$, $\sigma_2 = \sigma$, $\sigma_3 = 0$. According to the third theory of strength $\sigma_1 - \sigma_3 \leqslant [\sigma]$. Substituting $\sigma_1 = \sigma$ and $\sigma_3 = 0$ we obtain

$$\sigma = \frac{pr}{2\delta} \leqslant [\sigma],$$

i.e., the estimation of the strength is carried out as in the case of the uniaxial state of stress. According to the fourth theory of strength

$$\sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} \leqslant [\delta].$$

Since in this case $\sigma_1 = \sigma_2 = \sigma$ and $\sigma_3 = 0$, then

$$\sigma = \frac{pr}{2\delta} \leqslant [\sigma],$$

i.e., the same condition as according to the third theory of strength.

2. *Cylindrical vessel.*

In this case (Fig. 9.19*a*) $\rho_1 = r$ (radius of the cylinder) and $\rho_2 = \infty$ (the radius of curvature of the generatrix of the cylinder). From Laplace's equation we obtain

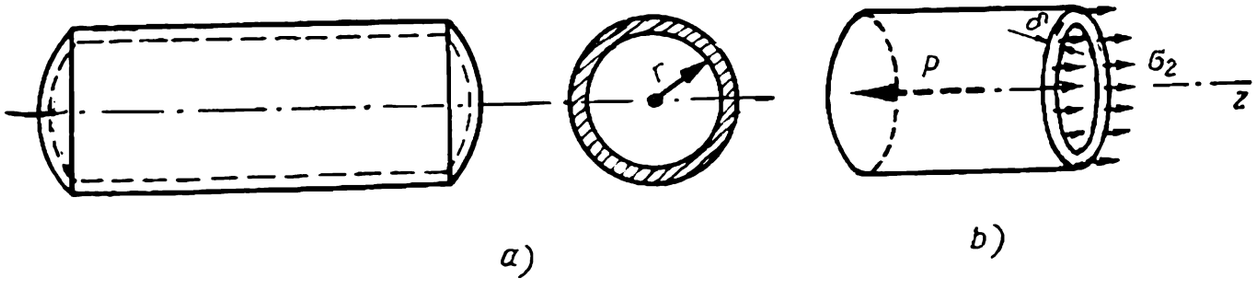
$$\frac{\sigma_1}{r} = \frac{p}{\delta},$$

* Examples of calculation of thin-walled vessels of other forms can be found in complete courses on strength of materials.

whence

$$\sigma_1 = \frac{pr}{\delta}. \tag{9.47}$$

To determine the stress σ_2 cut through the vessel in a plane perpendicular to its axis and consider the condition of equilibrium of one of the two parts of the vessel



a) b)

Fig. 9.19

(Fig. 9.19*b*). By projecting on the axis of the vessel all the forces acting on the cut part we obtain

$$-P + \sigma_2 2\pi r\delta = 0.$$

Here $P = \pi r^2 p$ is the resultant of gas pressure against the end of the vessel. Thus,

$$-p\pi r^2 + \sigma_2 2\pi r\delta = 0,$$

whence

$$\sigma_2 = \frac{pr}{2\delta}. \tag{9.48}$$

Note that since the wall thickness of the ring representing the transverse section of the cylinder acted upon by the stress σ_2 is small, its area is computed as the product of the length of the circumference and the thickness of the wall.

Comparing σ_1 and σ_2 in a cylindrical vessel we see that

$$\sigma_2 = \frac{1}{2}\sigma_1.$$

The condition of strength based on the third theory of strength for a cylindrical vessel is

$$\sigma_{eq} = \sigma_1 = \frac{pr}{\delta} \leqslant [\sigma]. \tag{9.49}$$

The condition of strength based on the fourth theory of strength is

$$\sigma_{eq} = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} \leqslant [\sigma].$$

Substituting the values of σ_1 and σ_2 from formulas (9.47) and (9.48) we obtain

$$\sigma_{eq} = 0.86 \frac{pr}{\delta} \leqslant [\sigma]. \tag{9.50}$$

The results obtained using formulas (9.49) and (9.50) differ by 14%. It is recommended that the formula based on the fourth theory of strength be used in this.

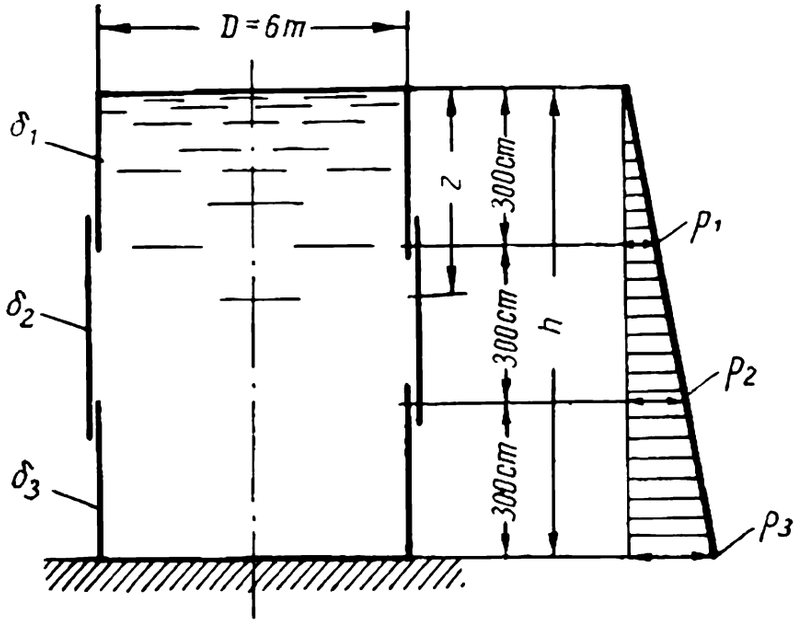


Fig. 9.20

Example 9.9. Determine with the aid of the fourth theory of strength the wall thickness of a cylindrical boiler of diameter $D=2$ m subjected to an internal steam pressure $p=10$ atm ≈ 10 kg/cm^2, $[\sigma]=1{,}000$ kg/cm^2.

Solution. From formula (9.50) we have

$$\delta = \frac{0.86pr}{[\sigma]} = \frac{0.86 \times 10 \times 100}{1{,}000} = 0.86 \text{ cm} = 8.6 \text{ mm}.$$

Assume $\delta = 10$ mm.

Example 9.10. Determine the wall thickness of a cylindrical reservoir for storing a fluid with specific weight $\gamma = 1$ t/m^3; the dimensions of the reservoir are indicated in Fig. 9.20. The allowable stress for the material of the walls is $[\sigma]=1{,}000$ kg/cm^2.

Solution. The fluid pressure against the walls of the vessel is proportional to the distance from the free surface:

$$p = \gamma z.$$

If the wall thickness of the vessel is constant, the analysis is carried out on the basis of the maximum pressure at the bottom:

$$p_{max} = p_3 = \gamma h = 1 \times 9 = 9 \text{ t/m}^2 = 0.9 \text{ kg/cm}^2.$$

The wall thickness according to formula (9.49) is

$$\delta_3 = \frac{0.9 \times 300}{1{,}000} = 0.27 \text{ cm} = 2.7 \text{ mm} \approx 3 \text{ mm}.$$

If the risk of corrosion is not too great to permit a smaller thickness, vessels of variable wall thickness should be employed; they should be broken into sections and the wall thickness of the sections determined for pressures p_1, p_2, and p_3 (at the bottom of each section).

CHAPTER X

STABILITY ANALYSIS OF COMPRESSED MEMBERS (BUCKLING)

85. Stable and Unstable Configurations of Equilibrium

From theoretical mechanics we know that equilibrium of an absolutely rigid body may be *stable*, *indifferent*, and *unstable*. For instance, a ball resting on a concave surface is in a condition of stable equilibrium. If it is given a small displacement from this position, it will return to its initial position (Fig. 10.1). A ball resting on a horizontal plane is in a condition of indifferent equilibrium. On being slightly displaced from this position it does not return to the initial position but its motion stops. Finally, a ball resting on a convex surface is in a condition of unstable equilibrium. Similar examples can be given from the sphere of equilibrium of deformable bodies.

Thus, a long column subjected to a comparatively small axial compressive force (less than a certain critical value) is in a condition of stable equilibrium (Fig. 10.2*a*). If it is deflected slightly by some lateral load and then the load is removed, the column will become straight again and will assume the original configuration of equilibrium. When the value of compressive force P equals the critical value P_{cr} the column will be in a condition of indifferent equilibrium. On being slightly displaced from the original straight position and left to itself, it will remain in equilibrium in this deflected position.* Finally, if force P exceeds the critical one, the straight configuration of equilibrium of the column will become unstable. At a load greater than critical a curvilinear configuration of equilibrium is stable but the column will now act in compression and bending rather than in central compression. Even the slightest excess of the force above its critical value leads to large

* Negligibly small deviations of the column from the straight position are considered here.

deflections of the column and to high stresses developed in it. The column either collapses or acquires inadmissibly large deformations. In both of these cases the column fails in practice, i.e., from the standpoint of engineering design the critical force must be considered to be the dangerous (ultimate) load.

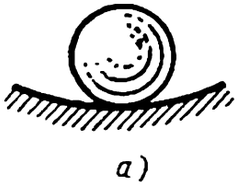


Fig. 10.1

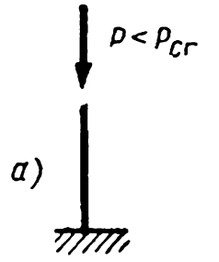


Fig. 10.2

Thus, *the critical force can be defined as a force at which a compression member is in a condition of indifferent equilibrium, in other words, as a force at which both the straight configuration of equilibrium of the member and a curvilinear configuration close to it (corresponding to infinitesimal deflections of the member) are possible.*

For safety's sake *the allowable load* must be naturally lower than the critical one:

$$[P] = \frac{P_{cr}}{n_s}, \qquad (10.1)$$

where n_{st} is *the stability factor of safety.*

The magnitude of the stability factor of safety is taken such that the reliability of the member is ensured despite the fact that the actual service conditions are likely to be less favourable than the conditions assumed in the design (because of non-homogeneity of materials, inaccuracy in determining the loads, etc.). The magnitude of the stability factor of safety is taken to be somewhat greater than the magnitude of the factor of safety in strength design, in order to allow for additional adverse circumstances such as initial curvature of a member, eccentricity of loading, etc.

For steel the magnitude of n_{st} is taken as being from 1.8 to 3, for cast iron from 5 to 5.5, for wood from 2.8 to 3.2. *

* The above values of the stability factors of safety are taken in structural design. The values of n_{st} as taken in the design of machine parts (such as guide screws of metal-cutting machines) are higher than the above, e.g., for steel n_{st} is taken from 4 to 5.

To take better account of concrete service conditions of compression members, it is recommended to employ a set of partial stability factors of safety rather than one general stability factor of safety, as is done in strength design.

Elastic instability is also possible in torsion, bending and under combined loading.

Investigations show that instability has been the cause of many accidents and structural damages.

86. Euler's Formula for the Critical Force

Consider a compression member in a critical condition when the compressive force has reached its critical value, and assume that the member is in a slightly deflected position (Fig. 10.3). If the moments of inertia with respect to two principal centroidal axes of the cross section are

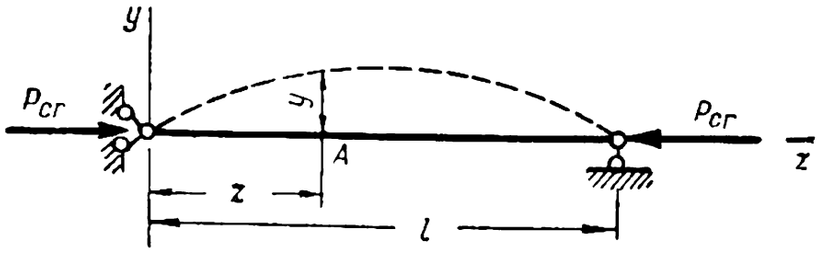


Fig. 10.3

not equal to each other, buckling will take place in the plane of the least rigidity, i.e., the cross sections of the member will rotate about the axis with respect to which the moment of inertia is a minimum. It can easily be demonstrated by applying a compressive load to a slender ruler.

In order to study buckling and to determine the critical force we shall use the approximate differential equation of the deflection curve of a beam (see Sec. 58):

$$EJ_{min}y'' = M_b. \tag{10.2}$$

The bending moment with respect to the centroid of the cross section A in the deflected position is

$$M_b = -P_{cr}y. \tag{10.3}$$

The minus sign is taken because the member bends concave downward and the ordinate y is positive. If the member

bended concave upward, the moment would be positive but the ordinate y would be negative and we would obtain the same result (10.3).

After substituting (10.3) equation (10.2) becomes

$$EJ_{\min}y'' = -P_{cr}y.$$

Designating

$$k^2 = \frac{P_{cr}}{EJ_{\min}}$$

we obtain

$$y'' + k^2y = 0.$$

This is a linear differential equation of the second order. Its general solution, as is known from mathematics, is

$$y = A\cos kz + B\sin kz. \qquad (10.4)$$

Here A and B are constants of integration to be determined from the known conditions at the ends of the member:

(1) at $z=0$ $y=0$,
(2) at $z=l$ $y=0$.

From the first condition we obtain $A=0$. Consequently, the member bends along a sine curve

$$y = B\sin kz.$$

From the second condition we obtain

$$B\sin kl = 0.$$

This relationship is valid in two cases.

Case 1: $B=0$. But if $A=0$ and $B=0$, then, as follows from equation (10.4), the deflections of the member are equal to zero, which is contrary to the initial assumption.

Case 2: $\sin kl=0$. This condition is satisfied when kl takes on the following infinite series of values:

$$kl = 0,\ \pi,\ 2\pi,\ 3\pi,\ \ldots,\ n\pi,$$

where n is an integer. Hence $k=\frac{n\pi}{l}$ and since

$$k = \sqrt{\frac{P_{cr}}{EJ_{\min}}}$$

then

$$P_{cr} = \frac{\pi^2 EJ_{\min}}{l^2}n^2.$$

We, thus, obtain an infinite number of values of the critical forces corresponding to different forms of buckling of the member.

From the practical point of view, only the smallest value of the critical force at which the member buckles is of interest.

The first root $n=0$ gives no solution of the problem. At $n=1$ we obtain the smallest value of the critical force:

$$P_{cr} = \frac{\pi^2 E J_{\min}}{l^2}. \tag{10.5}$$

This is Euler's formula. This force corresponds to buckling of the member along a sine curve with one half-wave:

$$y = B \sin \frac{\pi}{l} z.$$

Subsequent roots give greater values of the critical force and we shall not consider these. They correspond to buckling of the member along the sine curve with several half-waves, which occurs when buckling along the sine curve with one half-wave is prevented for some reason or other, for instance, because of the presence of intermediate constraints.

It should be noted that the constant B and so the shape of the deflected axis of the member have remained undetermined.

If the exact differential equation of the deflected axis is used instead of the approximate one in the study of buckling it proves possible to determine not only the value of the critical force but also the relation between the compressive force and the deflection of the member.

87. Effect of End Conditions of a Column

The ends of compression members are most frequently connected in one of the four ways shown in Fig. 10.4.

The second type—both ends are hinged—has been considered in the derivation of Euler's formula. For other end connections the formula for determining the critical load is

$$P_{cr} = \frac{\pi^2 E J_{\min}}{(\mu l)^2}, \tag{10.6}$$

where μ is the length coefficient which depends on the conditions at the ends of a column (Fig. 10.4), $l_m = \mu l$ is the

modified length of a column. The lesser μ, the greater the critical and, consequently, allowable load for a column. For instance, a column with both ends built-in can carry 16 times the load a column built-in at one end can. Therefore, both ends of a column should be fixed where possible. Though, it is not always practicable. The members to which the ends of a given column are attached are always more or less elastic, flexible. This introduces some indefiniteness in the design. For this reason, even if a column is attached to other members of the structure in such a manner that it

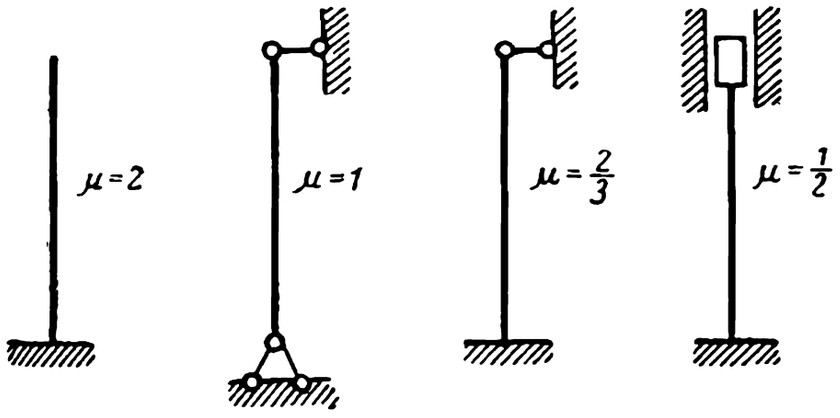


Fig. 10.4

can be considered to be a fixed-ended column, designers often keep on the safe side by assuming both ends to be hinged.

88. Limits of Applicability of Euler's Formula

Euler's formula is not always applicable. In its derivation we used the differential equation of the elastic curve which is based on Hooke's law. Hooke's law, however, is valid, as is known, so long as the stresses do not exceed the proportional limit.

In order to establish the limits of applicability of Euler's formula, let us determine the *critical stress* σ_{cr}, i.e., the stress which occurs at the cross section of a column under the critical load:

$$\sigma_{cr} = \frac{P_{cr}}{F} = \frac{\pi^2 E J_{min}}{(\mu l)^2 F}, \tag{10.7}$$

where F is the cross-sectional area of the column. But $i_{min} = \sqrt{\frac{J_{min}}{F}}$ is the least radius of gyration of the cross

section of the column. Therefore, formula (10.7) can be written in the form

$$\sigma_{cr} = \frac{\pi^2 E}{\left(\frac{\mu l}{i_{\min}}\right)^2}.$$

The quantity $\frac{\mu l}{i_{\min}}$ *characterises the effect of the dimensions of the column and end conditions; it is called the slenderness of the column and designated by* λ. The slenderness is a dimensionless quantity.

Thus, denoting

$$\lambda = \frac{\mu l}{i_{\min}}$$

we obtain

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2}. \tag{10.8}$$

For Euler's formula to be applicable, the following condition must be satisfied

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} \leqslant \sigma_p. \tag{10.9}$$

where σ_p is the proportional limit of the material of a column.

By writing formula (10.9) with respect to the slenderness ratio we obtain the condition of applicability of Euler's formula in the form

$$\lambda \geqslant \sqrt{\frac{\pi^2 E}{\sigma_p}}. \tag{10.10}$$

For example, for steel Ст. 3 $\sigma_p = 2{,}000$ kg/cm^2 and

$$\lambda \geqslant \sqrt{\frac{3.14^2 \times 2 \times 10^6}{2{,}000}} = 100.$$

Thus, *for columns of low-carbon steel Euler's formula is applicable if their slenderness is greater than* 100. In a similar manner we obtain the conditions of applicability of Euler's formula for cast iron $\lambda \geqslant 80$ and for wood $\lambda \geqslant 110$.

For medium and high-carbon steels as well as for alloy steels Euler's formula can be applied when the slenderness ratio is less than the above. Thus, for columns of chrome-molybdenum steel Euler's formula is applicable for $\lambda \geqslant 70$.

89. Empirical Formulas for Determining Critical Stresses

If, as this is often the case in practice, the slenderness of members is less than the above values, Euler's formula becomes inapplicable because critical stresses will exceed the proportional limit and Hooke's law will lose its validity.

There are approximate theoretical methods for determining critical inelastic buckling loads but they do not come within the scope of the present course.

In such cases use can be made of the following empirical formula obtained by Russian scientist F. S. Yasinsky on the basis of numerous experiments carried out by a number of investigators:

$$\sigma_{cr} = a - b\lambda, \tag{10.11}$$

where a and b are coefficients dependent on the material. For steel Ст. 3, for slenderness ratios from $\lambda=40$ to $\lambda=100$ the coefficients a and b may be taken to be $a=3{,}100$ kg/cm^2, and $b=11.4$ kg/cm^2.

For the slenderness ratios $\lambda<40$, members can be calculated on the basis of strength considerations without taking into account the danger of buckling.

90. Practical Formula for Stability Analysis

In place of two formulas, Euler's and Yasinsky's, each of which applies to a certain range of slenderness ratios, it is more convenient to have one formula that can be used whatever the slenderness.

This practical formula, which is widely employed in structural design, is

$$[P] = \varphi [\sigma_c] F. \tag{10.12}$$

Here $[\sigma_c]$ is the basic allowable compressive stress; φ is the reduction factor for the basic allowable stress (or the buckling factor). The magnitude of φ depends on the material and slenderness of a member and is determined from Table 10.1; F is the cross-sectional area of a member.

The quantity $\varphi[\sigma_c]$ may be regarded as the allowable stress in stability analysis, i.e.,

$$[\sigma_{st}] = \varphi [\sigma_c]. \tag{10.13}$$

Table 10.1

Slenderness ratio, λ	Steels: Ст. 1, Ст. 2, Ст. 3, Ст. 4	Steel Ст. 5	Improved steel $\sigma_y \geqslant 3{,}200$ kg/cm^2	Cast Iron	Wood
0	1.00	1.00	1.00	1.00	1.00
10	0.99	0.98	0.97	0.97	0.99
20	0.96	0.95	0.95	0.91	0.97
30	0.94	0.92	0.91	0.81	0.93
40	0.92	0.89	0.87	0.69	0.87
50	0.89	0.86	0.83	0.57	0.80
60	0.86	0.82	0.79	0.44	0.71
70	0.81	0.76	0.72	0.34	0.60
80	0.75	0.70	0.65	0.26	0.48
90	0.69	0.62	0.55	0.20	0.38
100	0.60	0.51	0.43	0.16	0.31
110	0.52	0.43	0.35	—	0.25
120	0.45	0.37	0.30	—	0.22
130	0.40	0.33	0.26	—	0.18
140	0.36	0.29	0.23	—	0.16
150	0.32	0.26	0.21	—	0.14
160	0.29	0.24	0.19	—	0.12
170	0.26	0.21	0.17	—	0.11
180	0.23	0.19	0.15	—	0.10
190	0.21	0.17	0.14	—	0.09
200	0.19	0.16	0.13	—	0.08

The basic allowable compressive stress $[\sigma_c]$, as is known, is given by [see formula (2.17)]

$$[\sigma_c] = \frac{\sigma_{\text{lim}}}{n}, \tag{2.17}$$

where σ_{lim} is the limiting stress taken to be the yield strength for ductile materials and the ultimate strength for brittle materials.

The relation among the factor φ, the critical stress σ_{cr}, the limiting stress σ_{lim}, the factor of safety n and the stability factor of safety n_{st} may be expressed as

$$[\sigma_{st}] = \varphi [\sigma_c] = \frac{\sigma_{cr}}{n_{st}}, \tag{10.14}$$

whence

$$\varphi = \frac{\sigma_{cr}}{[\sigma_c]\, n_{st}}. \tag{10.15}$$

By using formula (2.17), we obtain

$$\varphi = \frac{\sigma_{cr}}{\sigma_{lim}} \frac{n}{n_{st}}. \qquad (10.16)$$

For selecting a section, formula (10.12) is rearranged to give

$$F = \frac{[P]}{\varphi [\sigma_c]}.$$

The magnitude of φ has to be assigned, the slenderness ratio λ being unknown because the cross-sectional area F is unknown and λ depends upon this. As a first approximation it is recommended to take $\varphi_1 = 0.5$. Then the quantities F, J_{min}, i_{min}, λ are determined and the corresponding value of φ_1' is found from the table.

If the difference between the values of φ_1 and φ_1 appears to be great, the calculation should be repeated by assigning a new value φ_2:

$$\varphi_2 = \frac{\varphi_1 + \varphi_1'}{2}$$

and so forth until the difference between the subsequent values of φ does not exceed 4-6%.

For members whose cross sections have considerable weakenings (for instance, due to holes) in addition to stability analysis the usual strength analysis should be done as well, according to the formula

$$\sigma = \frac{P}{F_{net}} \leqslant [\sigma_c].$$

Here F_{net} is the working (net) cross-sectional area of a member. In the stability analysis, however, the total cross-sectional area (F_{gross}) is taken.

In some cases (for example, in the design of machine parts) the values of the stability factors of safety n_{st} allowed in the table of factors φ are inadequate. In such cases the calculations should be done with the direct use of the required factor n_{st} and either Euler's or Yasinsky's formula. The same procedure should be followed in the stability analysis of members made of materials which are not given in the table of factors φ.

91. Rational Shapes of Sections of Compression Members

Given a load, the length of a member, and the allowable stress, the shape and dimensions of the cross section of a compression member are characterised by the magnitude of the *radius of gyration*

$$i = \sqrt{\frac{J_{min}}{F}}.$$

The radius of gyration i is a dimensional quantity. For the purpose of comparing different cross sections it is more convenient to use the following dimensionless quantity

$$\rho = \frac{i}{\sqrt{F}} = \frac{\sqrt{J}}{F} = \frac{J}{F^2} \qquad (10.17)$$

which may be called the *unit radius of gyration.*

Table 10.2 gives the values of ρ for some of the most widely spread sections.

Table 10.2

Type of section	Magnitude of ρ
Rectangle with $\frac{h}{b} = 2$	0.204
Square	0.289
Circle	0.283
I-section	0.27-0.41
Channel	0.29-0.41
Angle	0.3-0.5
Ring with $c = 0.7$-0.9	1-1.2
Ring with $c = 0.8$-0.95	1.64-2.25

As is seen, rectangular solid sections are the least economical because the moments of inertia with respect to the principal axes are not equal and, consequently, the principle of equal stability of a member in both principal planes of inertia is not observed.

Ring and thin-walled box sections are the most economical. Estimates show that the replacement of angle and

I-compression sections by tubular members yields an economy of as much as 20-40% in material.

Example 10.1. Select an I-section for a compression hinge-ended member, if the compressive force $P=50$ t, and the length of the member is 2 m. The basic allowable stress is

$$[\sigma] = 1{,}600 \text{ kg/cm}^2.$$

Solution. Assuming as a first approximation $\varphi_1=0.5$ we obtain

$$F = \frac{P}{\varphi_1 [\sigma]} = \frac{50{,}000}{0.5 \times 1{,}600} = 62.5 \text{ cm}^2.$$

From the tables on profile sections (GOST 8239-56) we select I-section No. 36 for which $F=61.9$ cm^2. The least radius of gyration of the section from the same tables is $i_{\min}=i_y=2.89$ cm. The slenderness of the member is

$$\lambda = \frac{\mu l}{i_{\min}} = \frac{1 \times 200}{2.89} = 69.5 \approx 70.$$

The factor φ according to Table 10.1 for steel Ст. 3 at $\lambda=70$ is $\varphi'_1=0.81$. The difference between φ_1 and φ'_1 is considerable, therefore we repeat the calculation taking

$$\varphi_2 = \frac{0.5+0.81}{2} \cong 0.65.$$

Then

$$F = \frac{50{,}000}{0.65 \times 1{,}600} = 48 \text{ cm}^2.$$

We take I-section No. 30a for which $F=46.5$ cm^2, $i_y=2.69$ cm. The slenderness of the member is

$$\lambda = \frac{\mu l}{i_{\min}} = \frac{200}{2.69} \approx 75.$$

The factor φ from Table 10.1 (interpolating the values corresponding to $\lambda=70$ and $\lambda=80$) is $\varphi'_2 \approx 0.78$. The stress on the cross section of the member will be

$$\sigma = \frac{P}{F} = \frac{50{,}000}{46.5} = 1{,}080 \text{ kg/cm}^2.$$

The allowable stress in the stability analysis is

$$[\sigma_{st}] = \varphi'_2 [\sigma_c] = 0.78 \times 1{,}600 \cong 1{,}225 \text{ kg/cm}^2.$$

Understressing amounts to

$$\frac{1{,}225 - 1{,}080}{1{,}225} \times 100 = 11.8\%.$$

Make another attempt. Assume

$$\varphi_3 = \frac{0.78 + 0.65}{2} = 0.72.$$

We obtain

$$F = \frac{50{,}000}{0.72 \times 1{,}600} = 43.5 \text{ cm}^2.$$

We take I-section No. 27 for which $F = 40.2$ cm², $i_y = 2.54$ cm. The slenderness of the member is $\lambda = \frac{200}{2.54} = 79$). The factor φ from the tables is $\varphi_3' = 0.75$. The stress is

$$\sigma = \frac{50{,}000}{40.2} = 1{,}250 \text{ kg/cm}^2.$$

The allowable stress is $\varphi_3'[\sigma_c] = 0.75 \times 1{,}600 = 1{,}200$ kg/cm². Overstressing is $\frac{1{,}250 - 1{,}200}{1{,}200} \times 100 = 4.2\%$, which is permissible.

Example 10.2. Select the dimensions of a ring section for the same data as in the preceding example. Take the ratio of the inside diameter of the tube to the outside diameter to be $c = \frac{d}{D} = 0.9$. Compare the area of the ring section with the area of the I-section selected in the preceding example.

Answer: $D = 14.5$ cm, $d = 13.0$ cm, $F = 32$ cm². In comparison with the I-section, economy in the material amounts to $\frac{46.4 - 32}{46.4} \times 100 = 31\%$.

Example 10.3. Determine the allowable load of a compression member of steel Ст. 3 with a rectangular cross section 4×6 cm. The ends of the member are hinged.

The length of the member is $l = 80$ cm. The required stability factor of safety is $n_{st} = 3$.

Solution. Since a certain stability factor of safety is prescribed the calculations have to be done directly according to Euler's or Yasinsky's formula.

In order to decide which of the above formulas should be used we determine the slenderness of the member:

$$\lambda = \frac{\mu l}{i_{\min}},$$

where

$$i_{\min} = \sqrt{\frac{J_{\min}}{F}} = \sqrt{\frac{b^3 h}{12bh}} = \sqrt{\frac{b^2}{12}} = \sqrt{\frac{4^2}{12}} = 1.15 \text{ cm};$$

$$\lambda = \frac{1 \times 80}{1.15} = 69.5.$$

The slenderness ratio is less than 100, i.e., Euler's formula is not applicable for a member of low-carbon steel. Consequently, Yasinsky's formula should be used:

$$[P] = \frac{P_{cr}}{n_{st}} = \frac{\sigma_{cr} F}{n_{st}} = \frac{(a - b\lambda) F}{n_{st}} =$$

$$= \frac{(3{,}100 - 11.4 \times 69.5)\, 4 \times 6}{3} = 18{,}400 \text{ kg}.$$

92. Bending Induced by Axial Loading

Consider again the case of a member subjected simultaneously to a compressive force and a transverse load (Fig. 10.5). Under the action of this load the member de-

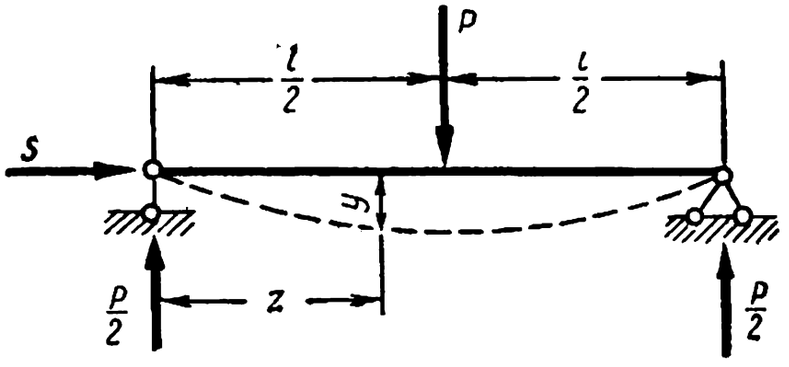


Fig. 10.5

forms, as shown by the dashed line in the figure. If the deformations are small as compared with the cross-sectional dimensions, the stresses developed in the member may be determined by using the principle of superposition. Those due to the compressive force acting separately are determined according to the formula

$$\sigma_1 = -\frac{S}{F}, \qquad (10.18)$$

while those due to the transverse load acting separately and causing bending—according to the formula

$$\sigma_2 = \pm \frac{M_{0b}}{W}, \tag{10.19}$$

where M_{0b} is the bending moment produced by the transverse load alone.

The total stress will, therefore, be determined by the known formula

$$\sigma = -\frac{S}{F} \pm \frac{M_{0b}}{W}. \tag{10.20}$$

For members with low rigidity it is necessary to take into account an additional bending moment produced at the cross section of the member by the axial force and equal, obviously, to $M_S = Sy$. In this case the stresses should be determined according to the formula

$$\sigma = -\frac{S}{F} \pm \frac{M_{0b} + M_S}{W}. \tag{10.21}$$

The design of a member of low rigidity under combined bending and compression, including additional bending moments due to the axial force and its effect on the deflections of the member, is usually known as design under bending induced by axial loading. When it is a question of bars of high rigidity under similar loading it is merely referred to as design under combined bending and compression.

The value of the deflection entering in the expression $M_S = Sy$ and, consequently, in formula (10.21) is not known. It cannot be determined by considering the transverse load alone. The additional bending moment produced by the compressive force also has to be taken into consideration. The magnitude of this moment, however, depends in turn on the deflection. Thus, the principle of superposition does not hold under bending induced by axial loading.

One of the methods of solving such problems is the method of successive approximations.

First the deflection that would be caused by the transverse load alone is found. For instance, for a beam resting on two supports and carrying a concentrated load at the centre the maximum deflection at the mid-span is (as shown previously)

$$y_{\max} = \frac{Pl^3}{48EJ}.$$

The maximum bending moment produced by this load occurs directly under the load and is given by

$$M_{0b} = \frac{Pl}{4}.$$

Taking this into consideration the preceding formula can be written in the form

$$y_{\max} = \frac{M_{0b} l^2}{12EJ}. \tag{10.22}$$

Knowing $y_{\max}$, it is possible to find the maximum bending moment due to the compressive force:

$$M_S = S y_{\max} \tag{10.23}$$

and the total bending moment

$$M_b = M_{0b} + M_S = \frac{Pl}{4} + S y_{\max}. \tag{10.24}$$

If the second term is not small compared with the first term, the solution must be made more exact. For that the value of M_b obtained should be substituted in formula (10.22) to find a closer approximation to the value of $y_{\max}$ and then more precise values of the bending moment and stress.

There exists a more exact solution of the problem of bending induced by axial loading which is treated in complete courses on strength of materials. This solution shows that deflection under bending induced by axial loading is determined by the formula

$$y = y_0 c, \tag{10.25}$$

where y_0 is the deflection caused by the transverse load alone, $c = \dfrac{1}{1 - \dfrac{S}{S_e}}$ is the correction factor accounting for the effect of the compressive force, S_e is the so-called Euler load. For a member with both ends hinged $S_e = \dfrac{\pi^2 EJ}{l^2}$.

The total bending moment is given by

$$M_b = M_{0b} + Sy = M_{0b} + \frac{Sy_0}{1 - \dfrac{S}{S_e}}. \tag{10.26}$$

The maximum stresses on the extreme fibres are determined according to the formula

$$\sigma = -\frac{S}{F} - \frac{M_b}{W} \tag{10.27}$$

or finally

$$|\sigma| = \frac{S}{F} + \frac{M_{0b}}{W} + \frac{Sy_0}{W\left(1 - \frac{S}{S_e}\right)}. \tag{10.28}$$

From formula (10.28) it follows that the stresses increase more rapidly than the load. In fact, if the transverse and axial loads are assumed to increase directly in proportion to a certain parameter, say, by a factor of n, then the magnitude of y_0 is also increased by a factor of n and the last term in formula (10.28) does not increase directly in proportion to n. Therefore strength design under bending induced by axial loading cannot be carried out on the basis of the allowable stresses. The calculations are done on the basis of the *allowable loads*, by determining the value of forces at which the stress at the dangerous point of the cross section reaches the yield point. By dividing this value of forces by the magnitude of the required factor of safety the allowable load is obtained. In checking the strength of a member the actual factor of safety is determined and compared with that required. Determination of the actual factor of safety is based on the assumption that formula (10.28) is valid up to the onset of yielding and that in the process of increasing the external loads the relation between the transverse and longitudinal forces remains unchanged. With these assumptions, by using formula (10.28) we may write

$$\sigma_y = \frac{Sn}{F} + \frac{M_{0b}n}{W} + \frac{Sny_0n}{W\left(1 - \frac{Sn}{S_e}\right)}. \tag{10.29}$$

Here n is the number indicating the factor by which the external forces must be increased to make the stress at the dangerous point equal to the yield point, in other words, n is the *actual factor of safety based on the loads.* From formula (10.29) n is found by solving the corresponding quadratic equation.

In cases when a transverse load produces bending in the plane of the greatest rigidity of a member, a check on

strength carried out as outlined above does not exclude the necessity of checking on the stability of the member in the plane of the least rigidity by the methods discussed in the preceding sections.

Example 10.4. Determine the maximum stress on the cross section of a simply supported beam. The cross section is I-section No. 20, with

$$F = 26.4 \text{ cm}^2, \quad W_x = 181 \text{ cm}^3, \quad J_x = 1{,}810 \text{ cm}^4.$$

The beam is loaded by force $P = 100$ kg at the mid-span and by the compressive force $S = 30$ t. The span of the beam is 8 m. The yield point is $\sigma_y = 2{,}400$ kg/cm^2.

Solution. The stresses due to the compressive force are

$$\sigma_1 = -\frac{S}{F} = -\frac{30{,}000}{26.4} = -1{,}135 \text{ kg/cm}^2.$$

The maximum bending moment due to the transverse load is

$$M_{0b} = \frac{Pl}{4} = \frac{100 \times 800}{4} = 2 \times 10^4 \text{ kg cm}.$$

The maximum normal stress due to this moment is

$$\sigma_2 = -\frac{M_{0b}}{W} = -\frac{2 \times 10^4}{181} = -110 \text{ kg/cm}^2.$$

The total stress, without taking into account the moment due to the longitudinal force, is

$$\sigma = -1{,}135 - 110 = -1{,}245 \text{ kg/cm}^2.$$

Now determine the stresses due to the additional moment produced by the longitudinal force. For that calculate the maximum deflection at the mid-span of the beam according to the formula developed previously

$$y_0 = \frac{Pl^3}{48FJ} = \frac{100 \times 800^3}{48 \times 2 \times 10^6 \times 1{,}810} = 0.296 \text{ cm}.$$

The Euler load is

$$S_e = \frac{\pi^2 EJ}{l^2} = \frac{3.14^2 \times 2 \times 10^6 \times 1{,}810}{800^2} = 56{,}400 \text{ kg} = 56{,}4 \text{ t}.$$

The correction factor is

$$c = \frac{1}{1 - \frac{30}{56.4}} = 2.14.$$

The additional stress due to the moment produced by the longitudinal force is

$$\sigma = \frac{-Sy_0c}{W_x} = \frac{30{,}000 \times 0.296 \times 2.14}{181} = -105 \text{ kg/cm}^2.$$

The total stress is

$$\sigma = -1{,}135 - 110 - 105 = -1{,}350 \text{ kg/cm}^2.$$

As is seen, the additional stresses due to the moment produced by the longitudinal force are comparatively small (8.4% of the total stresses) despite the fact that we have deliberately taken in the example a very flexible beam with the slenderness ratio $\lambda = \frac{l}{i_x} = \frac{800}{8.27} = 97$ and a considerable compressive force constituting 45% of the Euler load.

In calculating the safety factor use should be made of formula (10.29). We obtain $n = 1.5$. It means that if both the transverse and longitudinal loads are increased by a factor of 1.5, the maximum stresses will reach the yield point, i.e., 2,400 kg/cm^2. If we calculated the factor of safety based on the stresses, we would obtain another result:

$$n = \frac{2{,}400}{1{,}350} = 1.76.$$

However, if we increased the loads by a factor of 1.76, we would obtain, using formula (10.28), maximum stresses greatly exceeding the yield point of the material.

Thus, with bending induced by axial loading the factor of safety based on stresses does not give a correct idea of the degree of reliability of the structure, in view of the non-linear relation between loads and stresses. It is necessary, therefore, to calculate the safety factor based on loads.

In the example considered bending of the beam takes place in the plane of the greatest rigidity and it is therefore required to check on the stability of the beam in the plane of the least rigidity. In the stability analysis the moment of inertia should be taken as $J_y = 112$ cm^4.

CHAPTER XI

STRENGTH ANALYSIS UNDER DYNAMIC LOADS

93. Dynamic Loads

Up to this point we have studied the action of static loads on structural parts. As is known from the above, static loads change their magnitude from zero to their final values so slowly that accelerations obtained by structural elements are vanishingly small. After reaching their final magnitudes static loads subsequently remain practically constant. However, loads are very often dynamic in character. The action of such loads is accompanied by vibrations of structures and their individual elements.

Stresses developed as a result of vibrations of parts may be many times as great as stresses due to static loads.

The design of structural parts under dynamic loading is more complex than under static loading. The difficulty lies, on the one hand, in the more complicated methods of determining internal forces and stresses caused by a dynamic load and, on the other hand, in the more complicated methods of determining the mechanical properties of materials under dynamic loading.

For instance, under impact loading (i.e., loading of extremely short duration) many materials which are found to be ductile under static loading behave like brittle ones; under multiple-repeated alternating loadings the strength of materials is sharply reduced.

The general design method under dynamic loading is based on D'Alembert's principle, known from theoretical mechanics. According to this principle any moving body can be considered to be in a state of momentary equilibrium by adding to the external forces acting upon it an inertia force equal to the product of the mass of the body and its acceleration and counter to the acceleration. Therefore, in a case where inertia forces are known the method of

sections can be applied and the equations of equilibrium can be used to determine internal forces, without any restrictions.

In cases where the determination of inertia forces is difficult, as for instance under impact, dynamic stresses and deformations are obtained by using the law of conservation of energy.

94. Calculation of Stresses in Uniformly Accelerated Motion

In many cases accelerations in the movement of machine parts are known. Dynamic stresses are calculated without difficulty in these cases. Consider several examples.

Example 11.1. A body of weight Q is moved upward with acceleration a (Fig. 11.1). Determine the stress in the rope, neglecting its weight. We apply to the body an inertia force equal to $ma=\frac{Q}{g}a$ and acting downward. Use the method of sections. Make a cut a-a and remove the upper portion of the rope. Let N_d denote the force in the rope. Since the stresses in central tension are distributed uniformly over the section we may assume that $N_d=\sigma_d F$, where σ_d is the unknown dynamic stress in the rope.

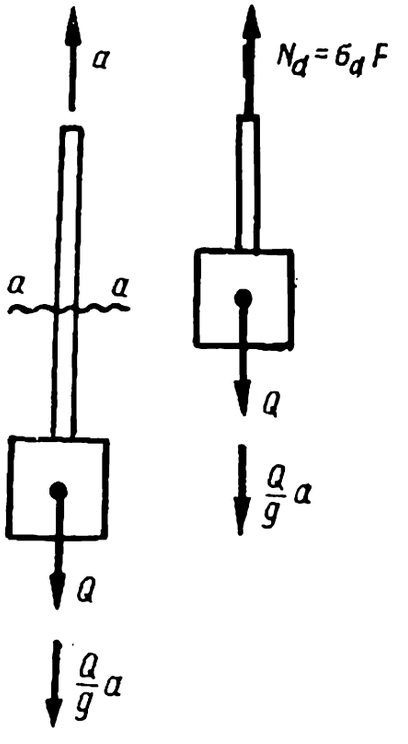


Fig. 11.1

Projecting all forces, including the inertia forces, on the vertical axis we obtain

$$\sigma_d F - Q\left(1+\frac{a}{g}\right)=0,$$

whence

$$\sigma_d=\frac{Q}{F}\left(1+\frac{a}{g}\right)=\sigma_s K_d.$$

Here $\sigma_s=\frac{Q}{F}$ is the stress under static loading, $K_d=1+\frac{a}{g}$ is a dynamic factor.

Thus, in many cases dynamic stresses can be expressed in terms of static stresses and a dynamic factor. It is especially convenient as the magnitude of a dynamic factor is often determined experimentally.

Example 11.2. A bar whose weight per unit length is q is lifted by means of two strings attached to its ends (Fig. 11.2). The motion is translatory with acceleration a. Determine the stresses in the bar.

Applying to each element of the bar of unit length the inertia force $\frac{q}{g}a$ we see that this problem is equivalent to the problem of a simple beam subjected to a uniformly distributed load of intensity $q+\frac{q}{g}a$.

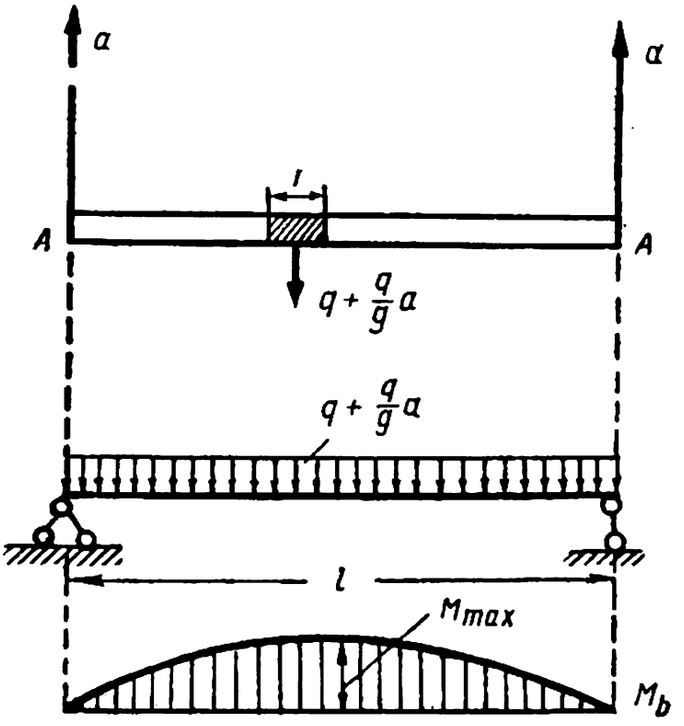


Fig. 11.2

The maximum bending moment occurs at the mid-section of the beam:

$$M_d=\frac{\left(q+\frac{q}{g}a\right)l^2}{8}=\frac{ql^2}{8}\left(1+\frac{a}{g}\right)=M_sK_d,$$

where $M_s=\frac{ql^2}{8}$ is the bending moment due to a static uniformly distributed load of intensity q; $K_d=1+\frac{a}{g}$ is a dynamic factor.

The maximum dynamic stress is determined by the usual flexure formula

$$\sigma_d=\frac{M_d}{W_x}=\sigma_sK_d.$$

95. Strength Analysis under Impact

Consider the case of a weight striking a fixed body. Let a body of weight Q fall from a height h upon a fixed bar (Fig. 11.3). The velocity of the body at the moment of impact is determined by the known formula of free falling:

$$v = \sqrt{2gh}.$$

This velocity will drop to zero during a very short time interval of impact measured in thousandth or hundredth

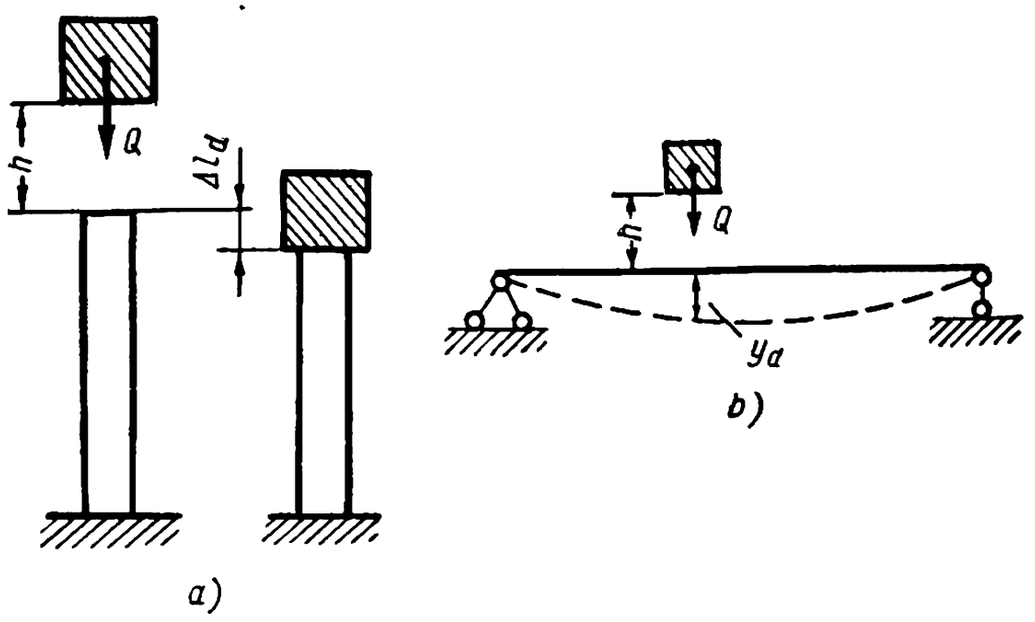


Fig. 11.3

parts of a second. The high degree of acceleration (deceleration) creates a considerable inertia force and it is the magnitude of this force that determines the impact action.

However, it is difficult to establish the law governing the variation of the velocity and, consequently, the magnitude of the inertia force theoretically. Another method is employed here, based on the law of conservation of energy and on the following assumptions.

1. The stresses developed under impact do not exceed the proportional limit, so that Hooke's law holds valid in impact loading.

2. The bodies do not separate from each other after impact.

3. The mass of the resisting bar is considered to be small in comparison with the mass of the striking body, therefore it is disregarded.

4. The loss of part of the energy dissipated in heat and in vibratory motion of the striking body is neglected.

Equate the kinetic energy of the falling weight to the potential energy of strain of the bar.

The kinetic energy is equal to the work done by the falling weight:

$$A = Q(h + \Delta l_d),$$

where Δl_d is the dynamic deformation (shortening) of the bar. The potential energy of strain in compression is (see Section 21)

$$\Pi = \frac{\Delta l_d^2 EF}{2l}.$$

We obtain from these two expressions.

$$Q(h + \Delta l_d) = \frac{\Delta l_d^2 EF}{2l}$$

or

$$\Delta l_d^2 EF - Q2l\,\Delta l_d - Qh\,2l = 0.$$

Dividing this equation through by EF we obtain

$$\Delta l_d^2 - \frac{Ql}{EF} 2\Delta l_d - \frac{Ql}{EF} 2h = 0.$$

But $\frac{Ql}{EF} = \Delta l_s$ is the deformation of the bar due to the statically applied load Q. Then

$$\Delta l_d^2 - 2\Delta l_s\,\Delta l_d - 2\Delta l_s h = 0.$$

Solving this quadratic equation with respect to Δl_d we obtain

$$\Delta l_d = \Delta l_s \pm \sqrt{\Delta l_s^2 + 2h\,\Delta l_s}.$$

Retaining the plus sign (a solution with the minus sign before the radical is contrary to the physical meaning of the problem) we obtain finally

$$\Delta l_d = \Delta l_s \left(1 + \sqrt{1 + \frac{2h}{\Delta l_s}}\right) = \Delta l_s K_d. \qquad (11.1)$$

Dividing both sides of the last equation by the length of the bar and multiplying by the modulus of elasticity E

we pass over, on the basis of Hooke's law, from deformations to stresses:

$$\sigma_d = \sigma_s\left(1 + \sqrt{1 + \frac{2h}{\Delta l_s}}\right) = \sigma_s K_d. \qquad (11.2)$$

It is seen from these formulas that the magnitudes of dynamic stresses and deformations depend on the magnitude of the static deformation, which depends in turn on the rigidity of the resisting body.

Shock-absorbers (rubber, spring) subject to large deformations are employed to alleviate shock.

In the case of a transverse (bending) impact the formulas have a similar form, only in this case Δl_s should be interpreted to mean the static deflection of a beam at the place of impact y_s and Δl_d the dynamic deflection y_d (Fig. 11.3*b*).

More exact investigations show that the foregoing approximate theory of impact is applicable (with an error of not more than 10%), if

$$\frac{2h}{\Delta' s} \leqslant 100.$$

Example 11.3. A weight $Q = 100$ kg falls from a height $h = 10$ cm onto the mid-span of a steel I-beam No. 27a having a span 3 m long. The moment of inertia of the cross section is $J_x = 5{,}500$ cm^4. The section modulus is $W_x = 407$ cm^3 (from tables on profile sections). $E = 2 \times 10^6$ kg/cm^2. Determine the maximum deflection of the beam and the maximum stresses developed on its cross section.

Solution. Calculate the static deflection of the beam below the weight according to formula (7.16):

$$y_s = \frac{Ql^3}{48EJ_x} = \frac{100 \times 300^3}{48 \times 2 \times 10^6 \times 5{,}500} = 0.0048 \text{ cm}.$$

Since y_s is considerably less than the height of the fall h we determine the dynamic factor according to the formula

$$K_d = \sqrt{\frac{2h}{y_s}} = \sqrt{\frac{2 \times 10}{0.0048}} = 63.$$

In this case the dynamic effect of the falling weight is 63 times its static effect.

Calculate the static stress due to the weight Q.

The maximum bending moment will occur at the mid-section of the beam. It is given by

$$M_b = \frac{Ql}{4} = \frac{100 \times 300}{4} = 7{,}500 \text{ kg cm.}$$

The maximum static stress is

$$\sigma_s = \frac{M_b}{W_x} = \frac{7{,}500}{407} = 18.6 \text{ kg/cm}^2.$$

The maximum dynamic stress is

$$\sigma_d = \sigma_s K_d = 18.6 \times 63 = 1{,}170 \text{ kg/cm}^2.$$

This example shows how dangerous dynamic loads are in their action. Furthermore the allowable stresses under impact are taken to be lower than under static loads.

96. Testing of Materials under Impact Loading (Impact Test)

Investigations show that the rate of straining has an appreciable effect on the mechanical properties of materials.

Fig. 11.4 shows two tensile test diagrams—under static loading (A) and under impact loading (B). It is seen from these diagrams that yield point and ultimate strength tend to be raised in impact tension. Investigations by N. N. Davidenkov and others show that the yield point is increased by 20-70% and the ultimate strength by 10-30% as compared with static tension. Ductility diminishes with an increase in the rate of straining. A tendency to brittle fracture can be observed at relatively slow rates of loading.

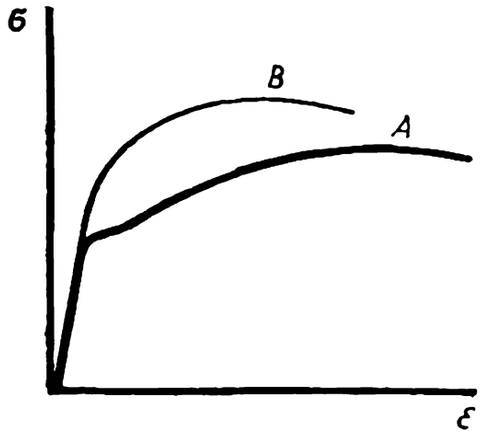


Fig. 11.4

To enable impact tensile test diagrams similar to diagram B in Fig. 11.4 to be constructed, special very complex machines are required. An alternative simplified method, the so-called *impact test*, is usually employed for estimating the properties of materials under impact loading. The standard form of specimen is used for testing. One such specimen is shown in Fig. 11.5.

At the mid-point of the specimen a 2 mm deep notch is cut to induce the material to work under the most severe stress conditions, as the notch gives rise to stress concentration.

The specimen is subjected to impact fracture on a special pendulum impact machine (Fig. 11.6). The knife of the

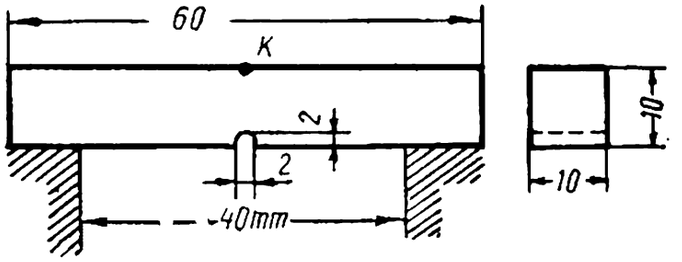


Fig. 11.5

pendulum (C), raised to height h_1, is let fall; it breaks the specimen, striking it at point K (see Fig. 11.5) and moving upward a distance $h_2 < h_1$ as a result of the remaining kinetic energy.

The work done by the pendulum is $A = Q(h_1 - h_2)$. Except for a small part ΔA, which is dissipated in overcoming adverse resistance (friction in the machine, air

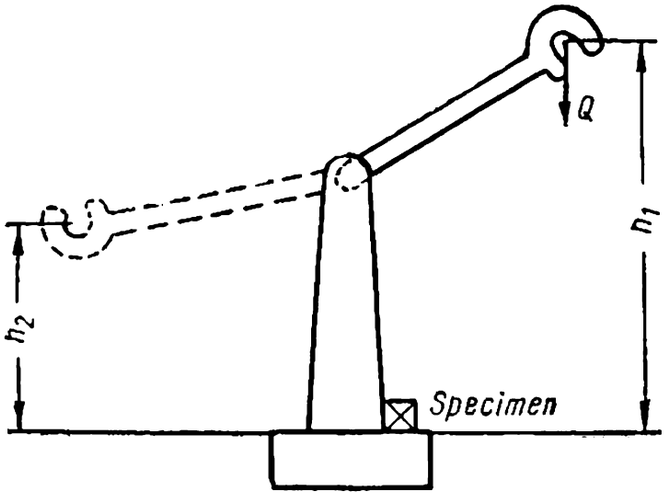


Fig. 11.6

resistance), it is expended on fracturing the specimen. The amount of these losses is known for each individual impact testing machine.

The following quantity is taken to characterise the ability of a material to resist impact loads

$$a_k = \frac{A_1}{F}\left[\frac{\text{kgm}}{\text{cm}^2}\right],$$

where $A_1 = A - \Delta A$ is the energy required to produce fracture, F is the cross-sectional area of the specimen at the point where the notch has been made.

The quantity a_k is called the unit notch toughness of a material. The greater a_k is, the better the material is for resisting impact and the more ductile it is.

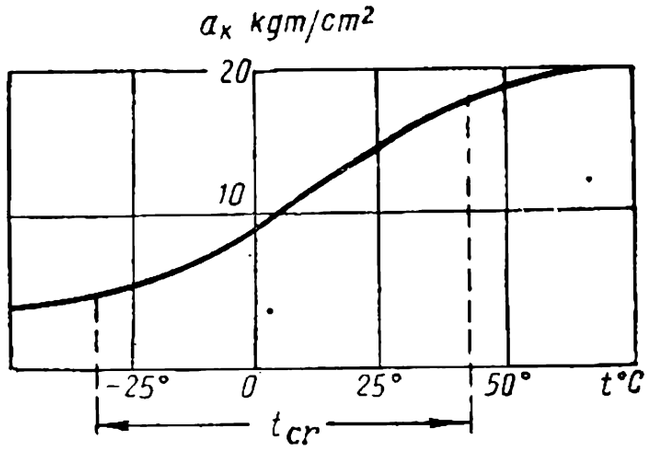


Fig. 11.7

The magnitude of the notch toughness a_k depends on the temperature t at which the test is conducted. For steel Ст. 3 the relation between a_k and t is plotted in Fig. 11.7.

Table 11.1

Steel	Heat treatment			
	Annealing		Quenching and temper	
	σ_u, kg/mm²	a_k, kgm/cm²	σ_u, kg/mm²	a_k, kgm/cm²
Mild steel C 0.15%	35-45	25	30-50	25
Steel C 0.20-0.30%	50-60	20	55-75	15
Steel C 0.30-0.40%	50-70	16	70-85	12
Steel C 0.50-0.60%	80-90	10	90-105	5
Nickel steel C 0.20%; Ni 3.0%	50-60	25-20	70-80	24-18

As the temperature decreases the magnitude of a_k diminishes. There exists a temperature interval t_{cr} when a_k diminishes especially rapidly. This interval is called *the critical temperature interval.*

The temperature range to the left of the critical interval is called *the range of temperature brittleness.* As can be observed, the range of temperature brittleness for steel Ст. 3 corresponds to temperatures below —25° C. At temperatures from —20° to +30° the magnitude of a_k for this steel is equal to 8-12 kgm/cm².

For some other grades of steel the values of notch toughness a_k at room temperature are given in Table 11.1.

97. Free Vibrations of a System with One Degree of Freedom

As has already been pointed out, a characteristic of dynamic loads is that they cause vibrations of structures and their individual members.

Consider the simplest problem of the theory of vibrations, the problem of the *free* (or natural) vibrations of a body whose mass is concentrated at one point (Fig. 11.8). *Vibrations are called free when they take place in the absence of external disturbing forces and are due only to the elastic properties of a vibrating system.*

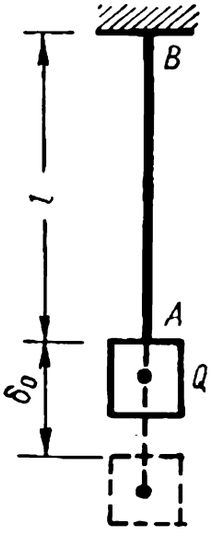


Fig. 11.8

Let us take the example of a weight Q suspended by an elastic weightless bar AB (or spring). To cause the body to vibrate about its equilibrium position let us elongate the bar AB by a certain amount δ_0 and leave the body to itself. It will start moving in a vertical direction. The position of the vibrating body is determined entirely by one parameter, the magnitude of δ, i.e., by the displacement of the body with respect to the position of static equilibrium. In this and similar cases it is said that a system has one degree of freedom.

The magnitude of displacement δ will vary with time according to a certain law. To establish this law use will be made of D'Alembert's principle. The vibrating body will be acted upon by the following forces:

1. The reaction on the part of the bar AB directed upward and given on the basis of Hooke's law by

$$N = \frac{EF\delta}{l}. \tag{11.3}$$

2. The upward inertia force given by

$$\frac{Q}{g}\frac{d^2\delta}{dt^2}. \tag{11.4}$$

The sum of these forces must be equal to zero

$$\frac{Q}{g}\frac{d^2\delta}{dt^2} + \frac{EF\delta}{l} = 0$$

or

$$\frac{d^2\delta}{dt^2} + \omega^2\delta = 0, \tag{11.5}$$

where

$$\omega = \sqrt{\frac{EFg}{Ql}} = \sqrt{\frac{g}{\Delta l_s}} \tag{11.6}$$

is the number of vibrations per 2π seconds called the *frequency of vibrations;* $\Delta l_s = \frac{Ql}{EF}$ is the static deformation of the bar AB. Equation (11.5) represents the differential equation of the free vibrations of the weight. It is an equation of the second order, homogeneous, linear, with constant coefficients. Its integral is

$$\delta = A\cos\omega t + B\sin\omega t. \tag{11.7}$$

It is seen from the equation obtained that the quantity ω represents the circular frequency of vibrations, i.e., it gives the number of complete vibrations per 2π sec.

The constants A and B are determined from the initial conditions. If at the initial moment, i.e., at $t=0$, the displacement of the body was $\delta=\delta_0$ and the velocity $v=v_0$, then from the equations

$$\delta_0 = A\cos 0 + B\sin 0,$$

$$\left.\frac{d\delta}{dt}\right|_{t=0} = v_0 = -\omega(A\sin 0 - B\cos 0)$$

we find

$$A = \delta_0, \quad B = \frac{v_0}{\omega},$$

and finally

$$\delta = \delta_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t.$$

Consider the following case of motion. A weight Q falling from height h strikes flange C, producing an extension of the bar CD (Fig. 11.9). Determine the further process of motion. The initial conditions are:

(1) at $t = 0$ $\delta = 0$,

(2) at $t = 0$ $v = v_0 = \sqrt{2gh}$.

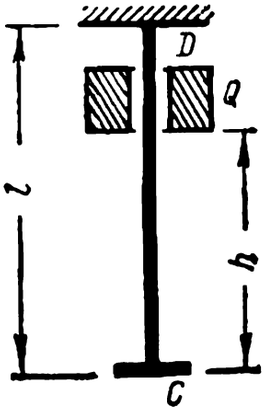


Fig. 11.9

Therefore,

$$A = 0, \qquad B = \frac{v_0}{\omega} = \sqrt{2h\,\Delta l_s}.$$

Thus, after impact the body will perform vibrations about the position of static equilibrium according to the law

$$\delta = \sqrt{2h\,\Delta l_s} \times \sin \omega t.$$

The frequency of the vibration of the weight is

$$\omega = \sqrt{\frac{EFg}{Ql}}.$$

The amplitude of the vibration is $\sqrt{2h\,\Delta l_s}$. The maximum displacement of the body (considering the static deformation) will be

$$\delta_{max} = \Delta l_s + \sqrt{2h\,\Delta l_s} = \Delta l_s \left(1 + \sqrt{\frac{2h}{\Delta l_s}}\right).$$

This result coincides with formula (11.1), which is derived in another manner (on the basis of the law of conservation of energy) *.

In the example considered above it was assumed that the free vibrations of a system governed by law (11.7) took place without energy dissipation, i.e., in the absence of resistance forces. On this assumption the vibrations will continue indefinitely long.

Actually, however, there always exist resistance forces of one sort or another (friction, air resistance, etc.) which

* If the term $\frac{2h}{\Delta l_s}$ is appreciably greater than unity.

lead to a continual reduction in the amplitude of free vibrations with the result that the vibrations will completely cease in a definite time interval.

The study of free vibrations in the presence of resistance forces does not lie within the framework of concise course on strength of materials.

98. Forced Vibrations of a System with One Degree of Freedom. Resonance

Consider now a case in which the system having one degree of freedom and subjected to vibration is acted upon not only by the reaction of the constraint and the inertia force but also by a disturbing force varying with time according to a certain law.

Let an unbalanced motor, say, a vibrator be attached to the weight Q (Fig. 11.8).

The law of variation of a disturbing force with time may be rather complicated.

Consider the frequently encountered case in which a disturbing force varies with time according to a sine law

$$P(t) = P_{\max} \sin \Omega t, \tag{11.8}$$

where $P_{\max}$ is the maximum value of the disturbing force, Ω is the frequency of the disturbing force.

The equation of motion of the weight (11.5) becomes

$$\frac{d^2\delta}{dt^2} + \omega^2\delta = P_{\max} \sin \Omega t. \tag{11.9}$$

The general integral of this non-homogeneous equation will be found as the sum of the integral of the homogeneous equation (i.e., without the right-hand member) and a particular solution of the non-homogeneous differential equation (11.9).

The integral of the homogeneous equation is

$$\delta = A \cos \omega t + B \sin \omega t. \tag{11.10}$$

A particular solution of equation (11.9) will be sought in the form

$$\delta^* = C \sin \Omega t, \tag{11.11}$$

where C is an undetermined coefficient.

To determine this coefficient we substitute solution (11.11) into the basic equation (11.9). We obtain (bearing in mind that $\frac{d^2\delta}{dt^2} = -C\Omega^2 \sin \Omega t$)

$$C = \frac{P_{max}}{\omega^2 - \Omega^2}. \qquad (11.12)$$

Thus the general integral of equation (11.9) is

$$\delta = A \cos \omega t + B \sin \omega t + \frac{P_{max}}{\omega^2 - \Omega^2} \sin \Omega t. \qquad (11.13)$$

As can be seen, the vibration of the weight is made up of the free vibration and the vibration due to the disturbing force. The latter vibration is called *forced vibration.* The frequency Ω of forced vibration, as is seen from (11.13), is equal to the frequency of the disturbing force (11.8).

The amplitude of forced vibration is

$$\frac{P_{max}}{\omega^2 - \Omega^2}.$$

If the frequency of the disturbing force approaches the frequency of free vibration of the weight in magnitude, the amplitude of forced vibration becomes very large.

When $\Omega = \omega$, however, the particular solution in the form of (11.11) is no longer valid as its substitution into equation (11.13) makes the left-hand member of this equation become zero.

In this case, i.e., when $\Omega = \omega$ a particular solution of equation (11.9) must be taken in the form

$$\delta^* = t [C \sin \Omega t + D \cos \Omega t]. \qquad (11.14)$$

Substituting this solution into (11.9), we obtain

$$C = 0; \qquad D = -\frac{P_{max}}{2\Omega}.$$

Thus the general solution of equation (11.9) is given by

$$\delta = A \cos \omega t + B \sin \omega t - \frac{tP_{max}}{2\Omega} \cos \omega t. \qquad (11.15)$$

As can be seen, the frequency of forced vibration is the same as that of free vibration but the amplitude $\frac{tP_{max}}{2\Omega}$ increases directly as the time.

The phenomenon of growth with time of the amplitude of vibration when the frequency of the disturbing force coincides with that of free vibration is known as *resonance*, and the coincidence of frequencies is called the *condition of resonance*. Although the resistance forces which are always present in reality restrain the growth of the amplitude of vibration, the phenomenon of resonance is highly dangerous and demands that all structures and their members be checked for resonance (when subjected to loads varying periodically with time).

If the frequencies of free and forced vibration are found to be close to each other, it is necessary to increase the difference by altering ω or Ω through structural measures.

If, however, the designer has no such possibility, it is necessary to resort to *damping* the structure, i.e., to apply special devices increasing the energy dissipation in vibration.

It should also be borne in mind that even in the case where $\omega=\Omega$ the occurrence of large-amplitude vibration requires some time for the system to be "stirred" [see formula (11.15)]. Therefore the short-term action of a disturbing force having the frequency $\Omega=\omega$ involves no danger as a rule.

More complicated problems of vibrations (vibrations of systems with several degrees of freedom, vibrations of beams, etc.) are treated in special textbooks.

CHAPTER XII

STRENGTH ANALYSIS UNDER ALTERNATING STRESSES (FATIGUE ANALYSIS)

99. Basic Definitions *

When periodically varying loads act for a long time another dangerous phenomenon occurs which is commonly called *fatigue of materials.*

The essence of this phenomenon is that parts which have been subjected to repeated loading over a long period of time fail at stresses lower than the ultimate strength and even the yield point. Fracture occurs as a result of the gradual propagation of a crack which is commonly termed a *fatigue crack.* The origin of the term "fatigue" is due to the erroneous assumption of the first investigators of this phenomenon that the structure of a metal changes under the action of alternating stresses.

At present it is ascertained that the structure of a metal does not change under the action of periodic loads. The nature of fatigue fracture is due to peculiarities of molecular and crystalline structure of substance. It seems to lie in non-homogeneity of the structure of materials. Individual crystallites of a metal have different strength in different directions. Consequently, plastic deformations occur in some crystallites at certain stresses.

Under repeated unloading and loading work-hardening takes place and the brittleness of a material increases. Eventually, after a large number of repetitions of stresses the ability of the material to harden is exhausted and a microcrack occurs along one of the slip planes of the crystallites. Once formed, the crack itself becomes a great stress raiser and owing to the progressive reduction in the

* There is no proper uniformity in the procedure of fatigue analysis as yet. We adhere to the procedure and notation used by S. V. Serensen (*Handbook of Mechanical Engineer.* Vol. 3. 1962).

cross-sectional area it becomes the focus of final fracture.

Two zones can be distinguished clearly at the fractured section: a zone with a smooth ground surface (zone of gradual propagation of the crack) and a zone with a rough surface (zone of final fracture due to reduction in the cross-sectional area).

Fig. 12.1 shows the photograph of the section of a fractured rail. An inner crack which has remained within the

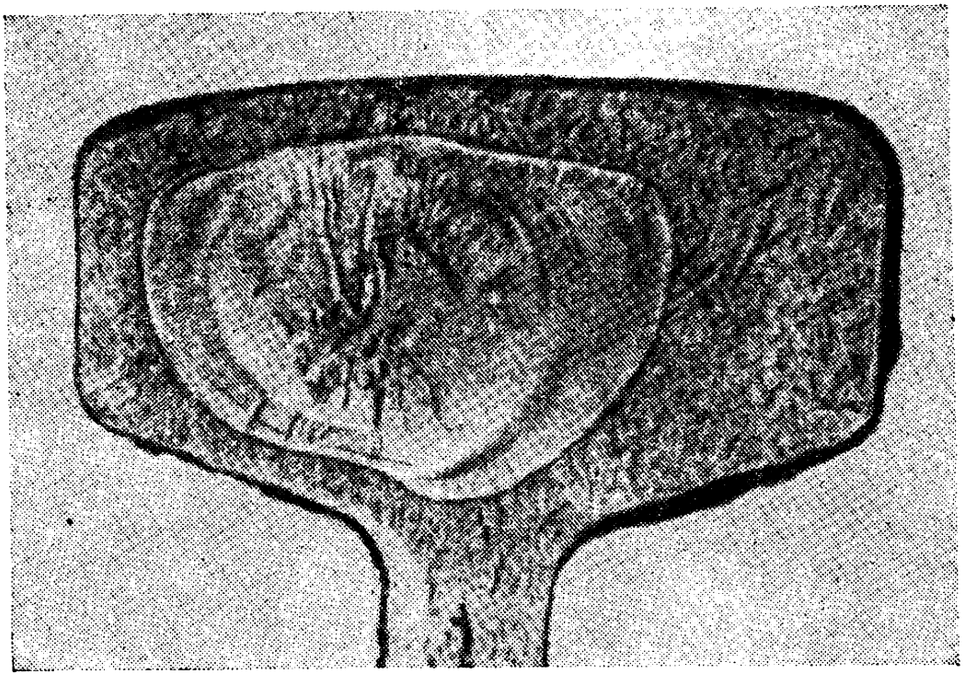


Fig. 12.1

rail after its rolling is enclosed by a smooth ground surface resulting from the gradual propagation of the crack and still farther by a rough surface of the section where the final fracture of the rail occurred because of a large reduction in its section.

Formation and propagation of fatigue cracks are essentially affected not only by defects of the internal structure of a material (inner cracks, slag inclusions, etc.) but also by defects of the surface finish of a part (scratches, cutting tool or grind-stone marks, etc.).

Investigations show that failures of machine parts are caused by fatigue cracks in the overwhelming majority of cases.

Let us discuss alternating loads, and so alternating stresses, in greater detail.

In general, as has already been noted in Sec. 4, loads and stresses may vary with time according to very complicated laws. Alternating stresses may have a steady and non-steady regime.

Under non-steady conditions the law of variation of stresses with time may be arbitrary in general.

Under steady conditions the variation of stresses with time is of a repeating (periodic) character. An exact repetition of stresses takes place in a definite time interval (period).

A set of all the values of the stresses during one period is called a *cycle*. It may also be said that a cycle is a single repetition of stresses.

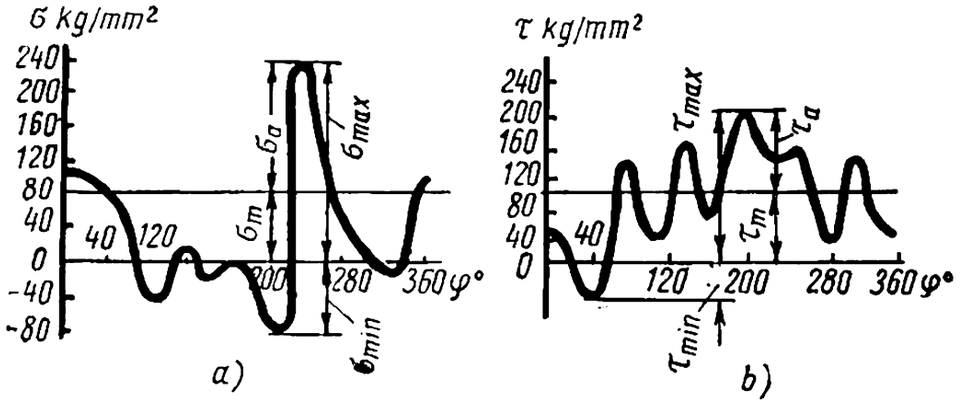


Fig. 12.2

Fig. 12.2 presents curves of variation in time of normal and shearing stresses in the crankshaft of a diesel engine during one revolution. As can be seen, the stresses vary according to a very complex law but are of a periodic (cyclic) character.

The effect of the shape of a curve of stress variation on the fatigue strength of parts has not been studied sufficiently, nevertheless the data available make it possible to conclude that this influence is insignificant, whereas the values of the maximum and minimum stresses of a cycle and their ratio play a dominant part. Therefore, in the following we shall assume that stresses vary in time according to a sine law, as shown in Fig. 12.3.

A cycle of alternating stresses (see Fig. 12.3*a*) is characterised by:

(1) maximum stress (algebraically) σ_{max};

(2) minimum stress σ_{min};

(3) mean stress $\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2}$; (12.1)

(4) amplitude of cycle $\sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2}$; (12.2)

(5) cycle ratio $r = \frac{\sigma_{min}}{\sigma_{max}}$. (12.3)

It is seen from formulas (12.1) and (12.2) as well as from Fig. 12.3 that

$$\sigma_{max} = \sigma_m + \sigma_a, \quad (12.4)$$

$$\sigma_{min} = \sigma_m - \sigma_a. \quad (12.5)$$

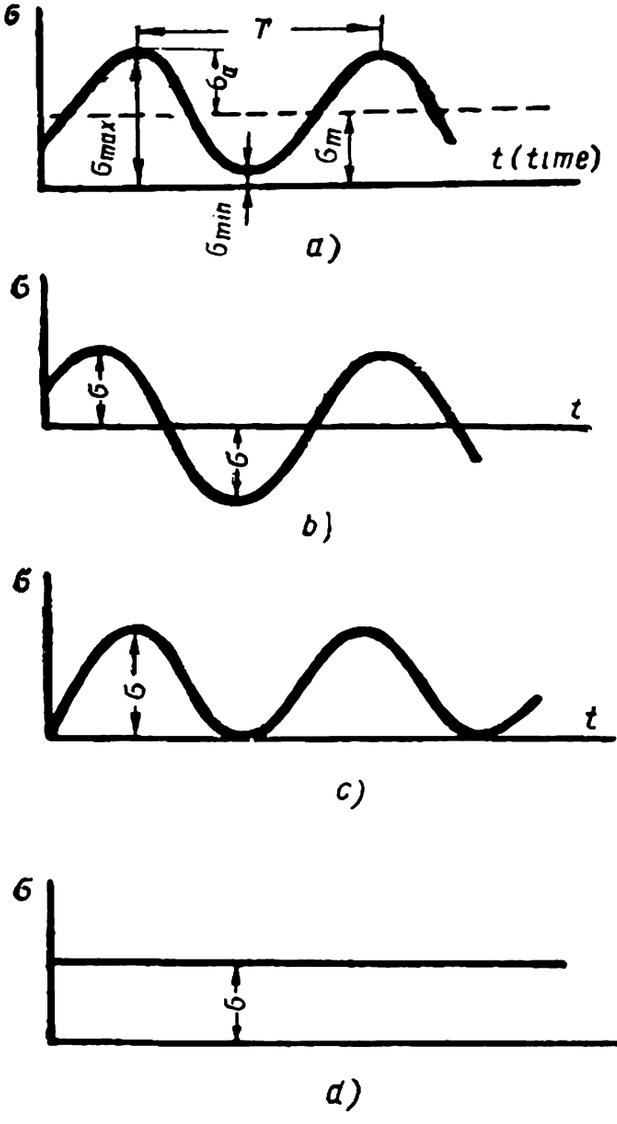


Fig. 12.3

If $\sigma_{max} = \sigma$ and $\sigma_{min} = -\sigma$, we have a *completely reversed cycle of stress* (Fig. 12.3*b*). In this case $\sigma_m = 0$, $\sigma_a = \sigma$, $r = -1$.

The cycle of stress shown in Fig. 12.3*c* is called *pulsating*. In this case

$$\sigma_{max} = \sigma, \quad \sigma_{min} = 0,$$

$$\sigma_m = \frac{\sigma}{2}, \quad \sigma_a = \frac{\sigma}{2}, \quad r = 0.$$

Constant static stress (Fig. 12.3*d*) can be considered as a particular case of the reversed cycle with the characteristics

$$\sigma_{max} = \sigma, \quad \sigma_{min} = \sigma,$$

$$\sigma_m = \sigma, \quad \sigma_a = 0, \quad r = \pm 1.$$

Any cycle of varying stresses can be represented as the sum of a completely reversed cycle with the maximum stress equal to the amplitude of the given cycle and the constant stress equal to the mean stress of the given cycle.

In the case of alternating shearing stresses all the foregoing terms and relations hold good if τ is substituted for σ.

100. Fatigue Curve for Complete Reversals of Stresses. Endurance Limit

Strength analysis under alternating stresses requires the knowledge of mechanical properties of materials. They are determined by testing specimens on special machines. The simplest and most commonly used tests are those under complete reversals of stresses. The principle of operation of one of the machines for testing specimens in bending is illustrated in Fig. 12.4.

A specimen *1* is clamped in the chuck *2* of the machine spindle rotating at a certain angular velocity. At the end of the specimen a ball-bearing *3* is mounted which transmits force P of constant direction. It can be easily seen

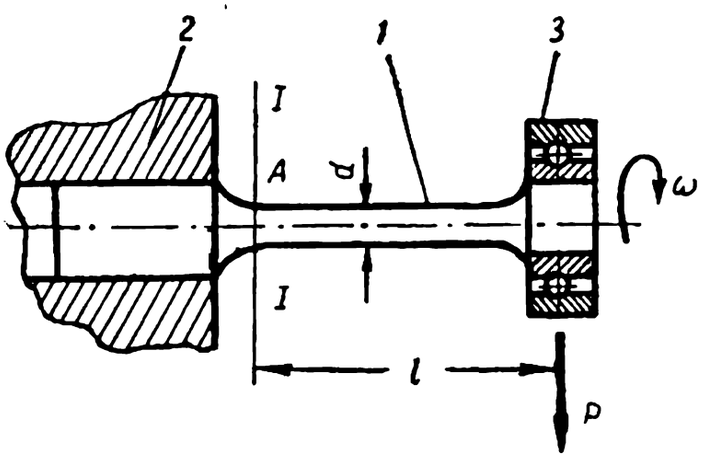


Fig. 12.4

that the specimen will be subjected to completely reversed bending stress. Indeed, the material in the specimen at the most dangerous point A undergoes tensile stress σ in the position indicated in Fig. 12.4 as the cantilever bends convex upward. However, after the specimen has rotated through half a revolution point A will be at the bottom, in the compression zone, and the stress at it will become $-\sigma$. After the next half revolution of the specimen point A will again be at the top and so on. When passing through the neutral axis the stress at point A is zero.

The test is conducted in the following succession, using 6-8 identical specimens, generally 6 to 10 mm in diameter with polished surface. The first specimen is loaded to a considerable stress σ_1 so that it will rupture after being subjected to a comparatively small number N_1 of revolutions (cycles). Concern is with the greatest stress of the

cycle for the most stressed point of the section. In bending, as is known, the greatest stress occurs at the outer points of the cross section and is determined by the formula

$$\sigma_{max} = \frac{M_b}{W_x}.$$

The test results are plotted in the co-ordinates $\sigma_{max} - N$ (Fig. 12.5).

After testing the first specimen point A is plotted whose co-ordinates are N_1 and σ_{1max} (or simply σ_1).

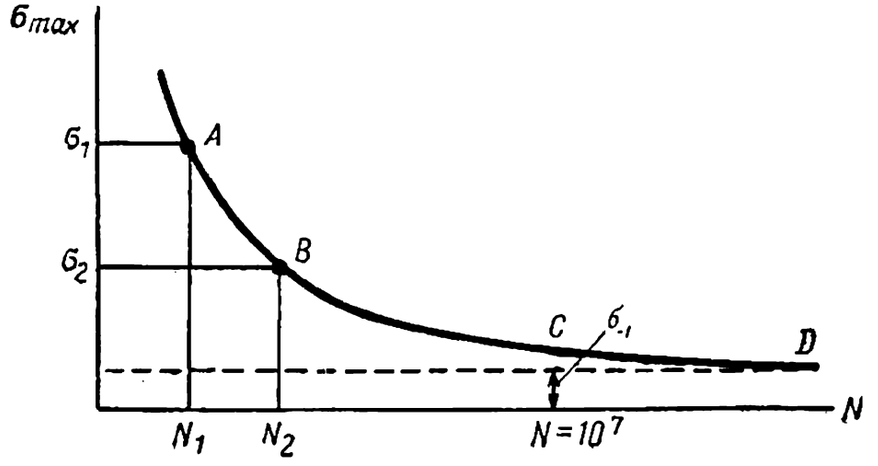


Fig. 12.5

A second specimen is then subjected to a slightly smaller stress σ_2. Naturally it will rupture after a larger number of cycles N_2. Point B is plotted with the co-ordinates N_2, σ_2, and so on.

Having tested all the specimens and joining points A, B, C, etc., by a smooth line, we obtain the curve $ABCD$ which is called the *fatigue curve* (or Wöhler's curve). The characteristic feature of this curve is that from a certain stress the curve is practically horizontal (portion CD). It means that at a certain stress the specimen can withstand an infinite number of cycles without rupturing.

The maximum value of the periodically varying stress which the material can resist indefinitely without rupturing is called the endurance (fatigue) limit.

As experiments show, a steel specimen which has not failed for 10^7 cycles (this number is called the base number of cycles) can withstand a practically infinite number of cycles. Therefore, after 10^7 cycles experiments with steel specimens are stopped.

The stress corresponding to $N=10^7$ is taken to be the endurance limit.

For non-ferrous metals the endurance limit is determined on the basis of 20×10^7 to 50×10^7 cycles. In a similar way, but using different machines, tests are conducted and endurance limits are established under axial forces (σ^0_{-1}), torsion (τ_{-1}) and combined stresses.

Today endurance limits for many materials are determined and given in handbooks. It is seen from these data that for most metals the endurance limit with completely reversed cycles of stress is less than the yield point.

Many machine parts are subjected to only a limited number of reversals of stress during their lifetime. In these cases the design is carried out on the basis of a higher, so-called *restricted endurance limit* at which the material withstands a given number of cycles. The magnitude of the restricted endurance limit is determined from the fatigue curve for a given number of cycles N.

101. Endurance Limit Diagram

To determine the endurance limit for any cycles of varying stresses another diagram is constructed which is called the *endurance limit diagram.* It may be constructed in several ways depending on which parameters of the varying stresses are taken as fundamental.

If the fundamental parameters of the varying stresses are taken to be the maximum stress of a cycle (σ_{max}) and the mean stress of a cycle (σ_m), the diagram is commonly called *Smith's diagram*; if, however, the fundamental parameters are taken to be the amplitude of stresses (σ_a) and the mean stress, Haigh's diagram is obtained.

Consider the construction of Smith's diagram. The maximum stress of a cycle is laid off along the vertical axis, the mean stress along the horizontal axis (Fig. 12.6).

First point C, whose ordinate represents the endurance limit σ_{-1} for a completely reversed cycle of stress, is plotted on the σ_{max} axis (for a completely reversed cycle the mean stress is equal to zero). Then the endurance limit is determined experimentally under some repeated loading, for instance, pulsating loading for which the maximum stress is always twice the mean stress.

Point P is plotted whose ordinate represents the endurance limit for a pulsating cycle σ_0.

For many materials the values of σ_{-1} and σ_0 have been determined and are given in handbooks.

In a similar way the endurance limit is determined experimentally for cycles of varying stresses with other parameters.

The results are plotted as points A, B, etc., whose ordinates represent the endurance limits for the corresponding cycles of stresses. Point D lying on the bisector OD will characterise the limiting stress (the ultimate strength) under constant loading for which $\sigma_{max} = \sigma_m$.

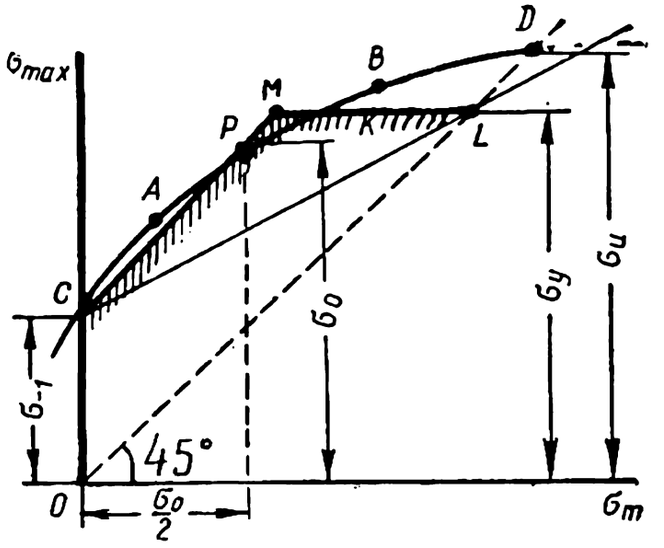


Fig. 12.6

Since for ductile materials the limiting stress in design is the yield point stress, a horizontal line KL is drawn on the diagram whose ordinate is equal to σ_y. (For ductile materials whose tensile test diagrams have no yield areas the yield strength $\sigma_{0.2}$ plays the part of σ_y.) Hence, the endurance limit diagram (to be more exact, the limiting stress diagram) will finally have the shape $CAPKL$.

This diagram is usually simplified by replacing it by two straight lines CM and ML, the straight line CM being drawn through point C (corresponding to a completely reversed cycle) and point P (corresponding to a pulsating cycle).

The above method of schematising the limiting stress diagram has been proposed by S. V. Serensen and R. S. Kinasoshvili.

In this case within the straight line CM the maximum stress of a cycle (the endurance limit) will be expressed by the equation

$$\sigma_{max} = \sigma_{-1} + \frac{\sigma_0 - \sigma_{-1}}{\frac{\sigma_0}{2}} \sigma_m = \sigma_{-1} + (1 - \psi_\sigma)\sigma_m \quad (12.6)$$

or

$$\sigma_a = \sigma_{max} - \sigma_m = \sigma_{-1} - \psi_\sigma \sigma_m, \quad (12.7)$$

where

$$\psi_\sigma = \frac{2\sigma_{-1} - \sigma_0}{\sigma_0}. \quad (12.8)$$

The factor ψ_σ characterises the sensitivity of a material to the cycle ratio. The values of ψ_σ and of the similar factor for torsion $\psi_\tau = \frac{2\tau_{-1} - \tau_0}{\tau_0}$ are given in Table 12.1.

Table 12.1

Factors	$\sigma_{u.t}$ in kg/mm²				
	32-42	40-50	50-75	70-105	105-125
ψ_σ (bending, tension, compression)	0	0	0.05	0.10	0.20
ψ_τ (torsion) . . .	0	0	0	0.05	0.10

In the absence of data concerning the magnitude of the endurance limit for a pulsating cycle a still more simplified diagram, in the form of straight line CL, is sometimes employed.

In fatigue analysis use is often made of the diagram σ_a-σ_m (Haigh's diagram) as well. For that the amplitude stress is laid off along the vertical axis and the mean stress along the horizontal axis (Fig. 12.7).

The working cycle of stresses in the part being designed is represented by point A with ordinate σ_a and abscissa σ_m.

Thus, all cycles having the same ratio $\frac{\sigma_a}{\sigma_m}$ (such cycles are called similar cycles) are represented by points lying on the straight line OA.

Having obtained from experiments for a given ratio $\frac{\sigma_a}{\sigma_m}$ the limiting values of the amplitude stress σ_a^{lim} and the mean stress σ_m^{lim} (corresponding to the endurance limit) a point B with the co-ordinates σ_a^{lim} and σ_m^{lim} is plotted on the straight line OA. The sum of the abscissa and the ordinate of this point gives the endurance limit for the corresponding cycle.

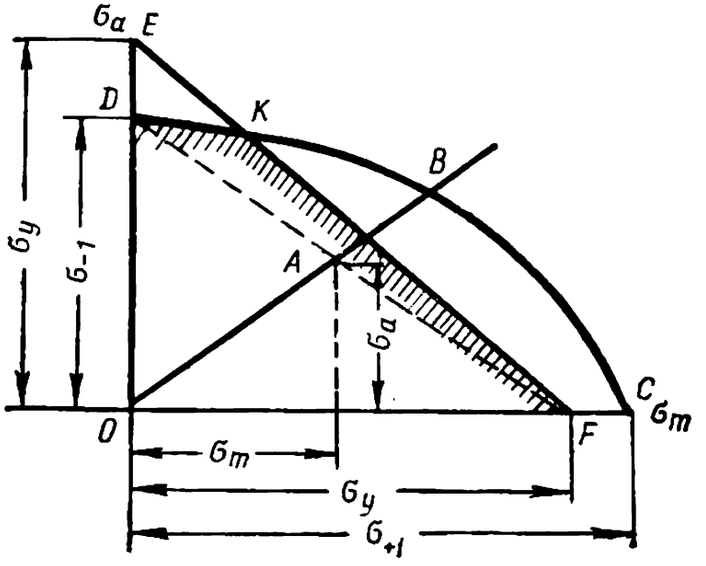


Fig. 12.7

Having carried out a series of tests with other characteristics of a cycle we obtain points corresponding to limiting cycles, i.e., cycles whose maximum stresses represent the endurance limits for cycles with different cycle ratios. By joining the above points we obtain curve DBC, the endurance limit diagram. As already noted, the magnitude of the endurance limit is determined by the sum of the co-ordinates of a point of curve DBC (see formula 12.4):

$$\sigma_r = \sigma_{\max}^{\text{lim}} = \sigma_m^{\text{lim}} + \sigma_a^{\text{lim}}.$$

Point C corresponds to constant stresses since in this case $\sigma_a = 0$. Point D corresponds to completely reversed stresses since $\sigma_m = 0$ for these.

For ductile materials the following correction should be introduced in this diagram. For such materials the limiting stress must not exceed the yield point:

$$\sigma_a + \sigma_m \leqslant \sigma_y.$$

Therefore, in the limiting stress diagram a straight line *EF* is plotted according to the equation

$$\sigma_a + \sigma_m = \sigma_y.$$

This straight line is equally inclined to the σ_a and σ_m axes. Finally the limiting stress curve has the shape *DKF*. The points lying above the line *DKF* characterise cycles for which either fatigue fracture takes place or plastic deformations occur. The points lying within the region *ODKF* correspond to safe cycles.

For the purpose of facilitating the design the limiting stress diagram is simplified and the segment *DK* is assumed to be straight.

Sometimes a still more simplified diagram is employed in the form of straight line *DF*. Calculations based on Smith's and Haigh's diagrams yield the same results under identical conditions.

102. Factors Affecting the Endurance Limit

Experiments show that the magnitude of the endurance limit is substantially affected by the following factors: stress concentration, the dimensions of a part, surface condition, the character of technological treatment, and some others. Consider these in greater detail.

A. Effect of Stress Concentration

Sharp changes in the form of a part, holes, grooves, notches, etc., reduce the endurance limit considerably as compared with that for smooth cylindrical specimens. This reduction is taken into account by *the effective stress concentration factor* which is determined experimentally. For that two series of identical specimens are taken (with 6 to 8 specimens in each series), the first without stress concentration and the second with stress concentration, and the endurance limits are determined with completely reversed cycles for specimens without stress raisers σ_{-1} and for specimens having some stress raiser σ_{-1k}.

The ratio

$$k_\sigma = \frac{\sigma_{-1}}{\sigma_{-1k}} \qquad (12.9)$$

determines the magnitude of the effective (actual) stress concentration factor. Experiments show that the effective stress concentration factor differs from the theoretical factor α_σ *, since the former depends not only on the shape of a part but on the material as well.

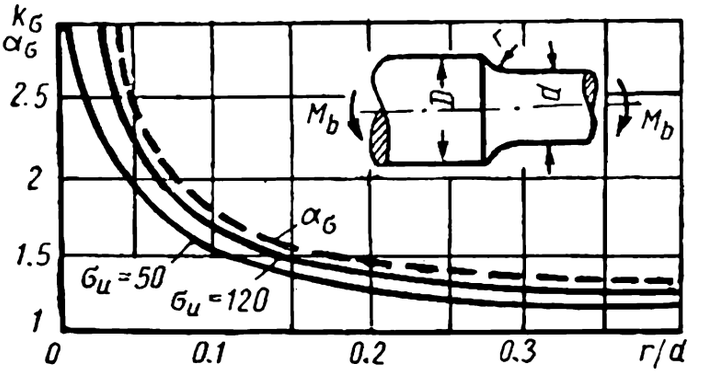


Fig. 12.8

The values of k_σ are given in handbooks. By way of an example, Fig. 12.8 gives the values of the effective stress concentration factors in bending for stepwise shafts having circular fillets with the ratio $\frac{D}{d}=2$. These data have been obtained in testing specimens having d=30-50 mm

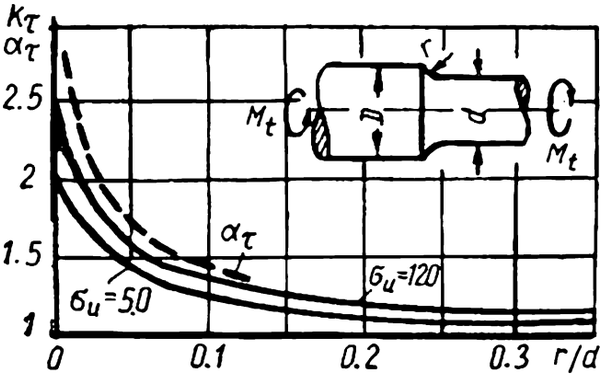


Fig. 12.9

for steels with the ultimate strength σ_u=50 kg/mm² and 120 kg/mm². For comparison the graph of the theoretical stress concentration factor α_σ is also given.

Fig. 12.9 gives the values of the stress concentration factors α_τ and k_τ in torsion and Fig. 12.10 gives them in

* The values of the theoretical factor are determined either theoretically (by the method of the theory of elasticity) or experimentally by comparing the ultimate strengths for specimens of highly brittle materials with and without stress concentration.

tension-compression. To determine the effective stress concentration factors for other ratios $\frac{D}{d}$ use can be made of the formula

$$k_\sigma = 1 + \xi [(k_\sigma)_0 - 1], \qquad (12.10)$$

where $(k_\sigma)_0$ is the effective stress concentration factor corresponding to the ratio $\frac{D}{d} = 2$, ξ is the correction factor determined from Fig. 12.11, curve *1* giving the values of ξ in bending and curve *2* in torsion.

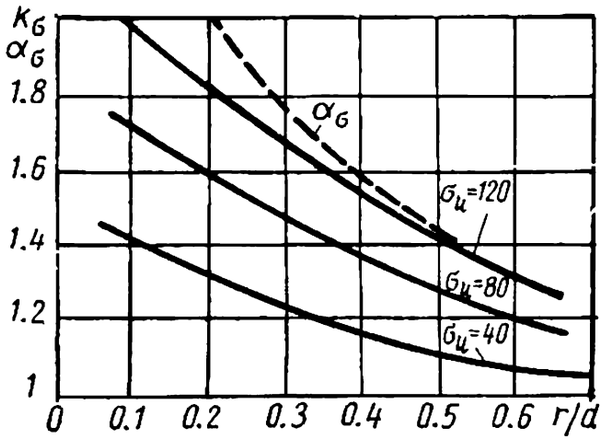


Fig. 12.10

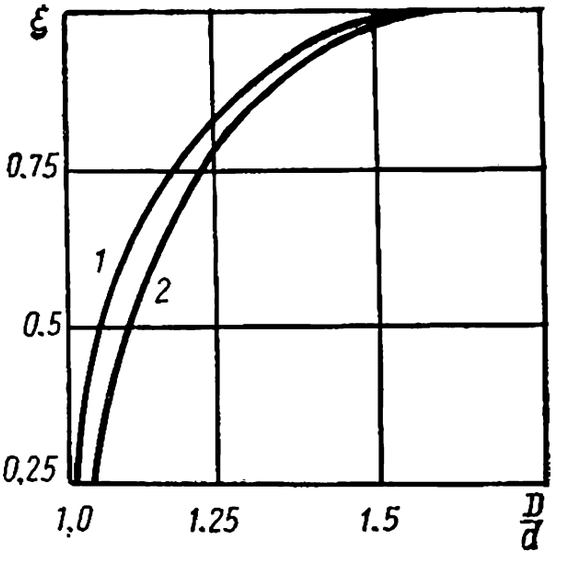


Fig. 12.11

The values of k_σ and k_τ for shafts with grooves (one or two) are given in Tables 12.2 and 12.3.

Table 12.2

$\sigma_{u.t'}$ kg/mm²	50	75	100
k_σ	1.5	1.75	2.0

Table 12.3

$\sigma_{u.t'}$ kg/mm²	60	70	80	90	100
k_τ	1.5	1.6	1.7	1.8	1.9

B. Effect of Absolute Dimensions of a Part

Experiments show that the larger the absolute dimensions of a part, the less the endurance limit. The ratio of the endurance limit of a part with dimension d to the endurance limit of a laboratory specimen of a similar configuration with small dimensions (d_0=6-12 mm) is

called the influence factor of the absolute dimensions of the cross section (or size factor):

$$\varepsilon_\sigma = \frac{(\sigma_{-1})_d}{(\sigma_{-1})_{d_0}} \quad \text{for normal stresses,}$$

$$\varepsilon_\tau = \frac{(\tau_{-1})_d}{(\tau_{-1})_{d_0}} \quad \text{for shearing stresses.}$$

The influence factors of the absolute dimensions of the cross section can be determined on the specimens with stress concentration as well. In this case

$$\varepsilon_{\sigma k} = \frac{(\sigma_{-1k})_d}{(\sigma_{-1k})_{d_0}},$$

$$\varepsilon_{\tau k} = \frac{(\tau_{-1k})_d}{(\tau_{-1k})_{d_0}}.$$

Both a part having dimension d and a specimen having dimension d_0 must have geometrically similar configuration.

Fig. 12.12 shows a graph of the values of ε_σ. Curve *1* corresponds to mild carbon steels with the ultimate ten-

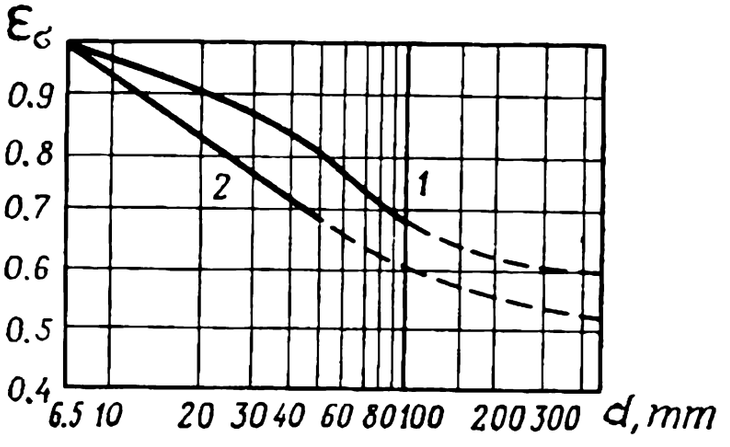


Fig. 12.12

sile strength $\sigma_{u.t} = 40$ to 50 kg/mm², curve *2* corresponds to high-strength alloy steels with the ultimate tensile strength $\sigma_{u.t} = 120$ to 140 kg/mm².

For intermediate values of the ultimate tensile strength it is necessary to make interpolation between the curves. Because of the lack of sufficient amount of experimental data concerning the factor ε_τ (in torsion) it may be assumed approximately that $\varepsilon_\tau \approx \varepsilon_\sigma$.

It should be noted that experimental data for determining ε are still sparse.

C. Effect of Surface Condition and Hardening of the Surface Layer

Experiments show that poor surface finish of a part reduces the endurance limit. The effect of surface finish is due to the change in the microgeometry and to the condition of the metal in the surface layer, which in turn depends on the method of mechanical treatment.

For estimating the effect of the surface condition on the endurance limit the factor β is introduced, which is equal to the ratio of the endurance limit for a part with a given surface finish (σ_{-1s}) to the endurance limit for a smoothly polished specimen (σ_{-1}):

$$\beta = \frac{\sigma_{-1s}}{\sigma_{-1}}.$$

Fig. 12.13 gives a graph of the values of β in relation to the ultimate strength σ_u of a steel and the type of surface finish. The curves correspond to the following types

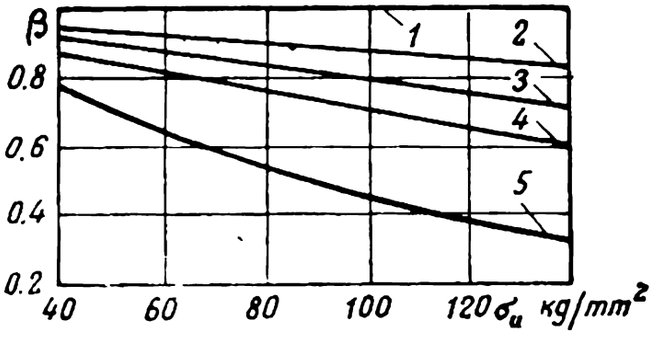


Fig. 12.13

of surface finish: *1*—polishing, *2*—grinding, *3*—fine turning, *4*—rough turning, *5*—presence of scale.

Various kinds of surface hardening increase the values of the factor β and it can reach values greater than unity.

Detailed information concerning the magnitude of β as influenced by the method of hardening of the surface layer is presented in handbooks.

103. Determination of Safety Factors for Completely Reversed Cycles of Stress

The safety factor is defined as the ratio of the endurance limit of a part to the maximum stress of a completely reversed cycle σ_a:

$$n_\sigma = \frac{(\sigma_{-1k})_d}{\sigma_a}. \qquad (12.11)$$

Here $(\sigma_{-1k})_d$ is the endurance limit of a part taking into account stress concentration, the effect of absolute dimensions and surface condition, etc.

$$(\sigma_{-1k})_d = \frac{\sigma_{-1}\varepsilon\beta}{k_\sigma}. \tag{12.12}$$

The safety factor in torsion is defined similarly:

$$n_\tau = \frac{(\tau_{-1k})_d}{\tau_a}. \tag{12.13}$$

Under combined stress the safety factor is usually calculated according to formula (9.44):

$$n = \frac{n_\sigma n_\tau}{\sqrt{n_\sigma^2 + n_\tau^2}},$$

where n_σ and n_τ are determined by use of formulas (12.11) and (12.13).

104. Determination of Safety Factor for any Cycle of Varying Stress

The simplified limiting stress diagram *CML* of a specimen (Figs. 12.6 and 12.14) is taken as a basis for calculation with any cycle of varying stress.

The limiting stress diagram of a part is constructed, with allowance for stress concentration, the effect of absolute dimensions, and surface condition. In accordance with test data the effect of the above factors is referred only to the variable component of a cycle, i.e., to the amplitude σ_a. The limiting stress amplitude for a specimen according to formula (12.7) is

$$\sigma_a = \sigma_{\max} - \sigma_m = \sigma_{-1} - \psi_\sigma \sigma_m. \tag{12.14}$$

The limiting stress amplitude for a part according to the foregoing is

$$(\sigma_{ak})_d = \frac{\sigma_a \varepsilon\beta}{k_\sigma} = \frac{(\sigma_{-1} - \psi_\sigma \sigma_m)\,\varepsilon\beta}{k_\sigma}. \tag{12.15}$$

The equation of the limiting stress line *EN* (see Fig. 12.14) for a part becomes

$$(\sigma'_{rk})_d = (\sigma'_{ak})_d + \sigma'_m = \frac{(\sigma_{-1} - \psi_\sigma \sigma'_m)\,\varepsilon\beta}{k_\sigma} + \sigma'_m, \tag{12.16}$$

where primes denote the current co-ordinates.

Now calculate the factor of safety for a part subjected to alternating stresses σ_{max} and σ_m (point R of the diagram—Fig. 12.14).

Assume that with an increase in the load the ratio $\frac{\sigma_{max}}{\sigma_m} = \text{const}$. Such loading is called *simple* loading. In this case the limiting point corresponding to failure will be point S.

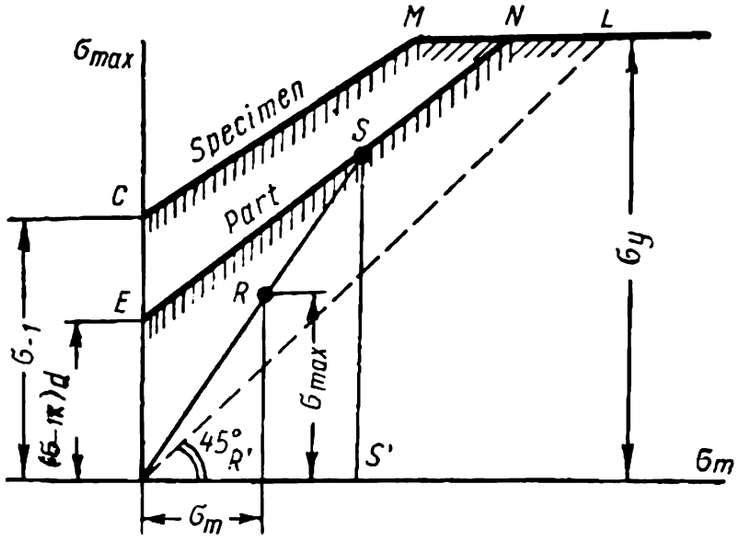


Fig. 12.14

The factor of safety will be equal to the ratio of the segments SS' and RR':

$$n_\sigma = \frac{SS'}{RR'} = \frac{(\sigma_{rk})_d}{\sigma_{max}}. \qquad (12.17)$$

The magnitude $(\sigma_{rk})_d$ (the ordinate of the point) will be found by solving the simultaneous equations of the line EN and the line OS. The equation of the line OS is

$$\sigma'_{max} = \frac{\sigma_{max}}{\sigma_m}\sigma'_m. \qquad (12.18)$$

Primes denote the current co-ordinates. Equating the right-hand members of formulas (12.16) and (12.18) we obtain

$$\frac{(\sigma_{-1} - \psi_\sigma \sigma'_m)\varepsilon\beta}{k_\sigma} + \sigma'_m = \frac{\sigma_{max}}{\sigma_m}\sigma'_m,$$

whence

$$\sigma'_m = \frac{\sigma_{-1}}{\psi_\sigma - \frac{k_\sigma}{\varepsilon\beta} + \frac{\sigma_{max}}{\sigma_m}\frac{k_\sigma}{\varepsilon\beta}}.$$

Substituting this value of σ'_m into formula (12.16) or (12.18) we find the ordinate to the point S:

$$(\sigma_{rk})_d = \frac{\sigma_{-1}\sigma_{\max}}{\frac{k_\sigma}{\varepsilon\beta}\sigma_a + \psi_\sigma\sigma_m}. \tag{12.19}$$

Hence, the following final relation for determining the safety factor is obtained on the basis of formula (12.17):

$$n_\sigma = \frac{\sigma_{-1}}{\frac{k_\sigma\sigma_a}{\varepsilon\beta} + \psi_\sigma\sigma_m}. \tag{12.20}$$

Likewise, in torsion

$$n_\tau = \frac{\tau_{-1}}{\frac{k_\tau\tau_a}{\varepsilon\beta} + \psi_\tau\tau_m}. \tag{12.21}$$

Under combined stress, for instance in torsion with bending, the factor of safety is calculated according to formula (9.44):

$$n = \frac{n_\sigma n_\tau}{\sqrt{n_\sigma^2 + n_\tau^2}},$$

whereas the values of n_σ and n_τ are given by formulas (12.20) and (12.21).

In addition to the factor of safety based on resistance to fatigue it is necessary to calculate that based on resistance to plastic deformations since point S may lie above the line ML. The safety factor based on resistance to plastic deformations is calculated according to the formulas:

$$n_\sigma = \frac{\sigma_y}{\sigma_{\max}} = \frac{\sigma_y}{\sigma_a + \sigma_m}, \tag{12.22}$$

$$n_\tau = \frac{\tau_y}{\tau_{\max}} = \frac{\tau_y}{\tau_a + \tau_m}. \tag{12.23}$$

The design (actual) factor of safety is the smaller of the factors as calculated according to formula (12.20) or (12.22); in torsion—according to formula (12.21) or (12.23). In the case of bending combined with torsion the smaller values of n_σ and n_τ calculated as outlined above should be substituted in the formula for determining the total safety factor (n).

105. Practical Measures to Combat Fatigue of Materials

In designing parts subjected to varying stresses it is recommended to take the following measures to increase fatigue strength.

1. Employ materials of maximum homogeneity with fine-grained structure, free from internal stress raisers (cracks, gas cavities, non-metallic inclusions, etc.).

2. Select parts so shaped as to eliminate stress concentration. Changes in cross section without transition curves should be avoided. In some cases it is recommended to employ special relieving grooves to reduce stress concentration. Thus, for instance, if near an abrupt change in cross section a smooth groove is made in a thicker portion of the part (Fig. 12.15) the localised stress will be sharply reduced.

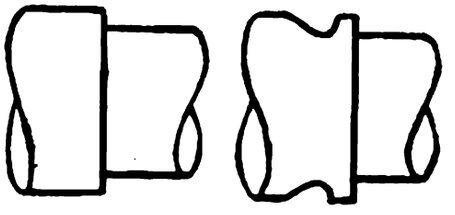


Fig. 12.15

3. Finish the surface of a part thoroughly, polishing it and removing the most minute scratches as these may give rise to fatigue cracks.

4. Employ special technological methods of increasing fatigue strength.

Such methods include:

(a) cold-working of the surface layer by cold rolling, shot peening, etc. In this way compressive stresses are set up artificially in the surface layer, decreasing the tensile stresses due to the load (orientated cold-working);

(b) chemical-thermal methods, such as case hardening, nitriding, cyaniding;

(c) thermal method—surface induction hardening or flame hardening.

Example 12.1. Calculate a shaft with a circular fillet of radius $r=5$ mm under repeated loading (Fig. 12.16).

The cross section of the shaft is acted upon by the bending moment $M_b=0.30$ tm and the twisting moment $M_t=0.40$ tm.

The material of the shaft is steel 45, $\sigma_u=6{,}100$ kg/cm^2, $\sigma_y=3{,}600$ kg/cm^2, $\sigma_{-1}=2{,}500$ kg/cm^2, $\tau_y=2{,}100$ kg/cm^2, $\tau_{-1}=1{,}500$ kg/cm^2.

The surface of the shaft is ground, hence the factor of surface quality $\beta=0.92$ (Fig. 12.13).

Assume that the bending stresses are completely reversed and the torsional stresses are pulsating.

Solution. Calculate nominal stresses

$$\sigma_{max} = \frac{M_b}{W_x} = \frac{0.3 \times 10^5}{0.1 \times 9^3} = 411 \text{ kg/cm}^2,$$

$$\tau_{max} = \frac{M_t}{W_p} = \frac{0.4 \times 10^5}{0.2 \times 9^3} = 275 \text{ kg/cm}^2.$$

Calculate the amplitudes and mean stresses of cycles

$$\sigma_m = 0, \quad \sigma_a = 411 \text{ kg/cm}^2, \quad \tau_m = \tau_a = 138.5 \text{ kg/cm}^2.$$

Determine the stress concentration factors. For $\frac{r}{d} = \frac{5}{90} = 0.055$ and $\frac{D}{d} = 2$ according to the graph in

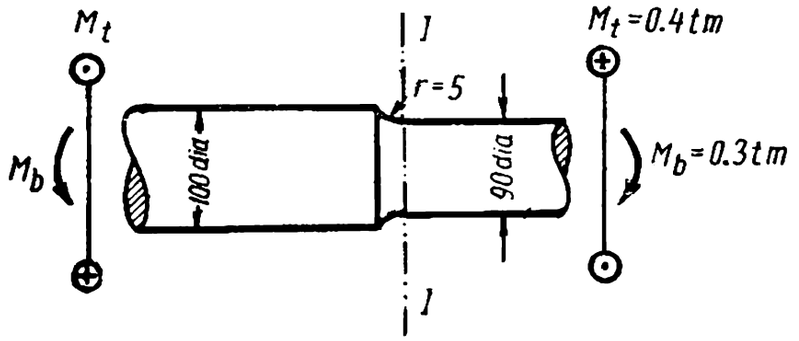


Fig. 12.16

Fig. 12.8 we have $(k_\sigma)_0 = 1.8$. Noting that in our case $\frac{D}{d} = \frac{100}{90} = 1.11$ and using the graph in Fig. 12.11 we find the correction factor $\xi = 0.7$. Then according to formula (12.10) we obtain

$$k_\sigma = 1 + 0.7(1.8 - 1) = 1.56.$$

The size factor for the shaft $d = 90$ mm is $\varepsilon_\sigma = 0.71$ from Fig. 12.12 (curve *1*).

Calculate the stress concentration factor in torsion. From the graph in Fig. 12.9 for $\frac{r}{d} = 0.055$ and $\frac{D}{d} = 2$ we have $(k_\tau)_0 = 1.4$. Taking into account that in our case $\frac{D}{d} = 1.11$, we find the correction factor $\xi = 0.58$ from the graph in Fig. 12.11 (curve *2*). Then according to formula (12.10)

$$k_\tau = 1 + 0.58(1.4 - 1) = 1.23,$$

The size factor in torsion is taken to be the same as in bending $\varepsilon = 0.71$.

Calculate the factors of safety:

(a) based on the endurance limit

$$n_\sigma = \frac{\sigma_{-1}}{\frac{k_\sigma \sigma_a}{\varepsilon\beta} + \psi_\sigma \sigma_m} = \frac{2{,}500}{\frac{1.56 \times 411}{0.71 \times 0.92} + 0} = 2.54,$$

$$n_\tau = \frac{\tau_{-1}}{\frac{k_\tau \tau_a}{\varepsilon\beta} + \psi_\tau \tau_m} = \frac{1{,}500}{\frac{1.23 \times 138.5}{0.71 \times 0.92} + 0} = 5.75,$$

$$n = \frac{n_\sigma n_\tau}{\sqrt{n_\sigma^2 + n_\tau^2}} = \frac{2.54 \times 5.75}{\sqrt{2.54^2 + 5.75^2}} = 2.35.$$

(b) based on the yield point

$$n'_\sigma = \frac{\sigma_y}{\sigma_{max}} = \frac{3.600}{411} = 8.7,$$

$$n'_\tau = \frac{\tau_y}{\tau_{max}} = \frac{2{,}100}{275} = 7.65,$$

$$n' = \frac{8.7 \times 7.65}{\sqrt{8.7^2 + 7.65^2}} = 5.7.$$

As can be seen, the minimum factor of safety is that based on the endurance limit $n = 2.35$. It should be compared with the normative factor of safety $n = n_1 n_2 n_3$ whose magnitude is established in accordance with the considerations outlined in Sec. 12.

Assuming the accuracy of determining the stresses to be sufficiently high, we may take $n_1 = 1.3$.

Assuming an ordinary level of technology, an average degree of homogeneity of the material and average dimensions of a part, we take $n_2 = 1.5$.

Assuming an average degree of responsibility of a part, we take $n_3 = 1.3$.

Then the total magnitude of the normative factor of safety is given by

$$n = 1.3 \times 1.5 \times 1.3 = 2.53.$$

Thus in this example the actual factor of safety ($n = 2.35$) is somewhat less than the normative factor of safety, hence it is necessary either to increase the diameter or to take a stronger steel.

CHAPTER XIII

RESISTANCE OF MATERIALS TO PLASTIC DEFORMATIONS. LIMIT DESIGN

106. Design Diagrams for an Elastic-Plastic Material

Up to this point we have used Hooke's law in deriving design equations.

From tension (compression) test diagrams we have seen that Hooke's law is valid only when stresses do not exceed the proportional limit. Tolerating some inaccuracy we used Hooke's law up to stresses equal to the yield point. However, the attainment of the yield point at a single point, even though the most dangerous one, does not yet mean collapse of a part or the appearance of deformations causing failure. Owing to plastic deformations and strain hardening of the material the less stressed particles of the material are brought into action thus making it possible to increase the allowable load on the structure.

The permanent set of 0.2 per cent assumed in establishing the magnitude of the yield point is not ultimate and in many cases can be increased.

In order to make calculations with allowances for plastic deformations of the material it is necessary to establish the design stress-strain relation (σ-ε diagram).

This diagram must be as simple as possible and at the same time it must reflect the most important properties of materials.

The real σ-ε diagrams can be divided into three types (Fig. 13.1):

(a) the diagram in Fig. 13.1*a* is characteristic of brittle materials (cast iron, brittle casting);

(b) the diagram with a pronounced yield point is characteristic of highly ductile materials (low-carbon steels) (Fig. 13.1*b*);

(c) the diagram in Fig. 13.1*c* is characteristic of a steel of low or medium ductility as well as of non-ferrous met-

als and their alloys of both medium and high ductility.

For brittle materials the deviation from Hooke's law is relatively small and in designing parts of such materials plastic deformations, because of their smallness, can be neglected, assuming that the material obeys Hooke's law up to the moment of failure (Fig. 13.1*d*). In designing parts of highly ductile materials (Fig. 13.1*b*) the design diagram including plastic deformations is taken as shown in Fig. 13.1*e*, that is strain hardening of the material is

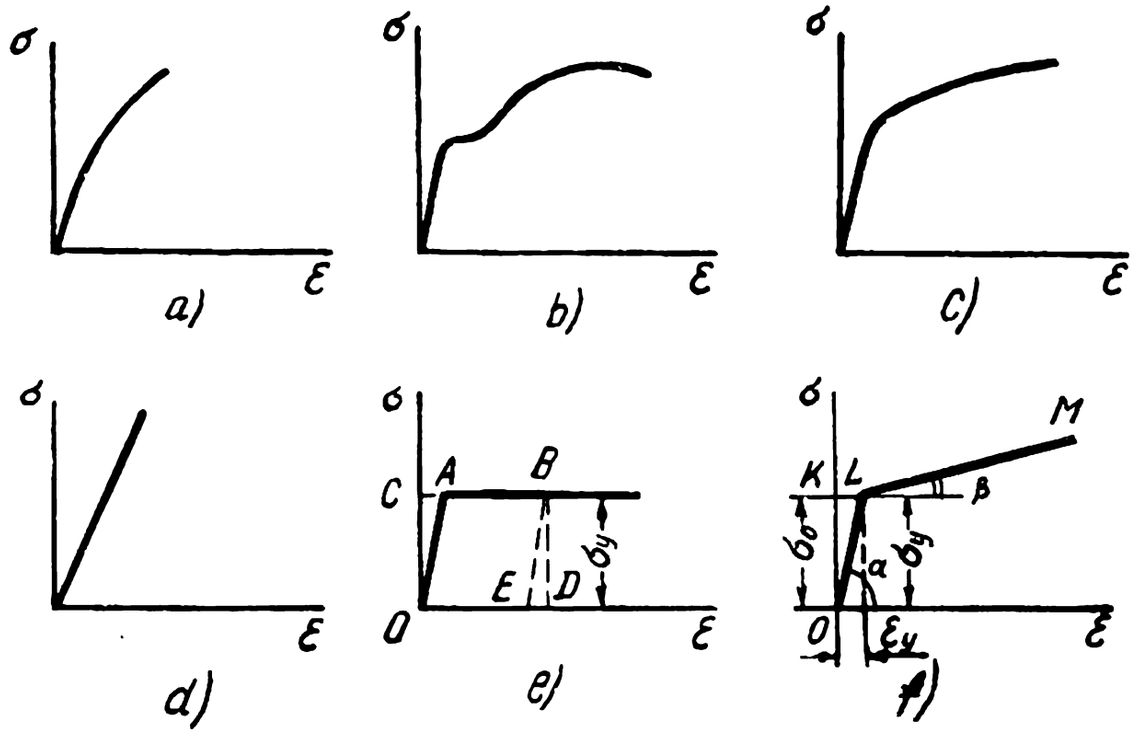


Fig. 13.1

disregarded since at stresses equal to the yield point considerable plastic deformations develop and displacements occur which make the part unserviceable. Sometimes a still more simplified diagram *OCB* is applied, i.e., the material is considered to be perfectly plastic. With such a simple assumption concerning the properties of a material it is possible to solve the most complex problems of plasticity: combined stress, statically indeterminate systems.

Below consideration is given to certain problems, the simplest ones, which are solved by use of this type of diagram.

For materials of medium ductility the design diagram of elastic-plastic deformations is taken as shown in

Fig. 13.1*f*. It takes into account strain hardening of a material.

Up to point *L* the material is assumed to deform elastically according to Hooke's law, beyond point *L* it is assumed to deform plastically (straight line *LM*). Some investigations have been carried out using the diagram *OKM*.

It is essential to note that in the region of plastic deformations the laws of loading and unloading are different, the law of loading being of a non-linear character and the law of unloading being linear at all times. In Fig. 13.1*e* the law of loading is characterised by the line *OAB*, the law of unloading by the line *BE*, always parallel to the line *OA*. The total strain *OD* consists of two parts: plastic *OE* and elastic *ED*, which disappears after the removal of the load.

It should also be noted that in a study of plastic deformations, just as in an investigation of elastic deformations the phenomena observed fall into two categories:

Category 1: small elastic-plastic deformations when deformations are small as compared with the dimensions of the bodies under strain;

Category 2: large elastic-plastic deformations when the change in the dimensions of a body, in the distance between the points of application of forces, etc., cannot be neglected.

Below we deal with cases of small elastic-plastic deformations. The discussion is limited to static single loadings since under reloadings it is necessary to determine plastic deformations occurring in a part after each unloading. A part subjected to repeated (alternating) loading must be designed on the basis of fatigue considerations.

107. Plastic Design of Statically Indeterminate Systems Acting in Tension-Compression

Consider the example from Section 14 (Fig. 13.2).

When solving this example on the assumption that deformations are elastic, i.e., stresses are within the proportional limit, we have found

$$N_1 = N_3 = 0.4P; \quad N_2 = 0.2P.$$

Since the bars have the same cross-sectional area, the stress developed in the outer bars $\sigma_1 = \sigma_3 = \frac{0.4P}{F}$ will be

twice as large as in the middle bar $\sigma_2 = \frac{0.2P}{F}$. The dangerous load * ($P_y$) will be determined by equating the stresses in the outer bars to the yield point stress

$$P_y = \frac{\sigma_y F}{0.4} = 2.5\sigma_y F. \qquad (13.1)$$

Examine now the behaviour of the system with a further increase in the load, assuming the material to be perfectly plastic (see Fig. 13.1*e*).

In spite of an increase in the load the forces in the outer bars will not be raised, but will remain equal to $\sigma_y F$.

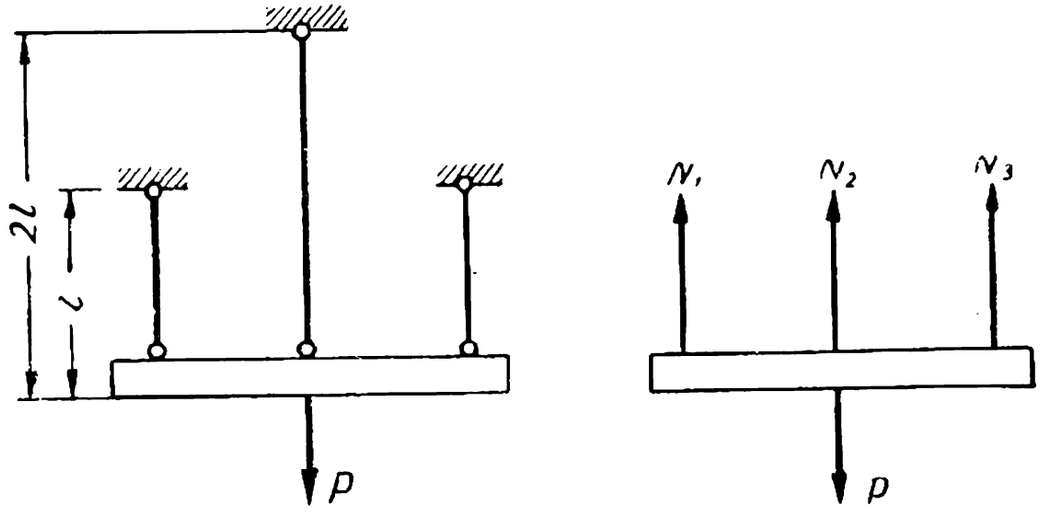


Fig. 13.2

The force in the middle bar, however, will increase until stresses in it reach the yield point. When the forces in all three bars have become equal to $\sigma_y F$, the load-carrying capacity of the system will be exhausted. From the condition of equilibrium of the system in the limiting state we obtain

$$P_u = 3\sigma_y F. \qquad (13.2)$$

Comparing this expression with expression (13.1) we observe that when plasticity of the material is taken into account the ultimate load is found to be greater than the dangerous load in the design based on elastic deformations.

As a second example consider the system represented in Fig. 13.3. Determining the forces in the bars of this system

* In design within the elastic range the dangerous load is one at which stresses reach the yield point at least in one bar. Of course, the term "dangerous" is of a relative character.

on the assumption of the elastic behaviour of the material we have established that the most highly stressed bar is the middle one (Sec. 14).

It is the yielding of the material in this bar that controls the dangerous load of the system:

$$P_y = \sigma_y F (2 \cos^3 \alpha + 1).$$

In designing this system on the basis of plastic deformations the limiting state of the system should be the one in which the stresses in all three bars reach the yield point and the forces reach a value of $\sigma_y F$.

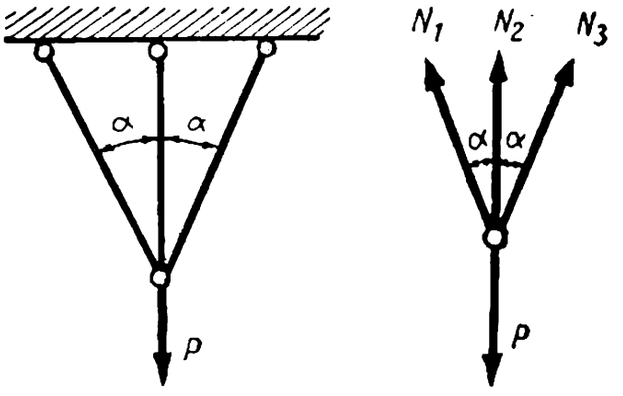


Fig. 13.3

From the condition of equilibrium of the cut portion we obtain

$$\sum Y = 0, \quad \sigma_y F + 2\sigma_y F \cos \alpha = P_u.$$

The ultimate load is given by

$$P_u = \sigma_y F (2 \cos \alpha + 1) > P_y.$$

It follows from this—and this is a general rule—that *when plastic deformations are taken into account the ultimate load of a statically indeterminate system is found to be greater than in the design based on the elastic behaviour.* For this reason under static loading we are in certain cases in a position to increase the allowable load as given by

$$[P] = \frac{P_u}{n}.$$

In the design based on the elastic behaviour

$$[P] = \frac{P_y}{n'}.$$

The ratio of the allowable loads determined by the above two methods is not equal to the ratio $\frac{P_u}{P_y}$ since the factor of safety n', as a rule, is taken 15-20% higher than n. Thus, the gain in the magnitude of the allowable load (or in the weight of the material—in selecting sections) when it is determined on the basis of plastic deformations is somewhat smaller than one would think in comparing the dangerous (P_y) and ultimate (P_u) loads.

It should be noted that for a statically determinate system whose elements work in tension (compression) the values of P_y and P_u coincide.

108. Plastic Torsion of a Circular Bar

In studying the elastic torsion of a circular bar (Sec. 35) the maximum stresses at the contour points of the cross section were found to be

$$\tau_{max} = \frac{M_t}{W_p}. \tag{13.3}$$

The magnitude of the dangerous twisting moment will be obtained from the condition that the maximum stresses should be equal to the yield point stress (Fig. 13.4*a*):

$$M_{ty} = \tau_y W_p = \tau_y \frac{\pi d^3}{16} \approx \tau_y 0.2 d^3. \tag{13.4}$$

However, this moment is not ultimate for a ductile material. A further increase in the twisting moment is possible owing to the growth of stresses at internal, less stressed points of the cross section.

The limiting state for a perfectly plastic material will be the one in which the stresses at all points of the cross section become equal to the yield point stress (Fig. 13.4*b*).

Let us write down the expression for the ultimate twisting moment as the resultant moment of the internal shearing forces occurring at the cross section (in the limiting state):

$$M_{tu} = \int_F \tau_y \, dF\rho. \tag{13.5}$$

Since $dF=2\pi\rho d\rho$ (the area of the elementary ring) (Fig. 13.4*c*)

$$M_{tu}=2\pi\tau_y\int_0^r\rho^2\,d\rho, \tag{13.6}$$

or

$$M_{tu}=\frac{2}{3}\pi r^3\tau_y=W_{ppl}\tau_y. \tag{13.7}$$

Here

$$W_{ppl}=\frac{2}{3}\pi r^3=\frac{\pi d^3}{12}. \tag{13.8}$$

The quantity W_{ppl} is called the plastic section modulus in torsion.

Comparing formulas (13.7) and (13.4) we see that when plasticity of the material is taken into account the ulti-

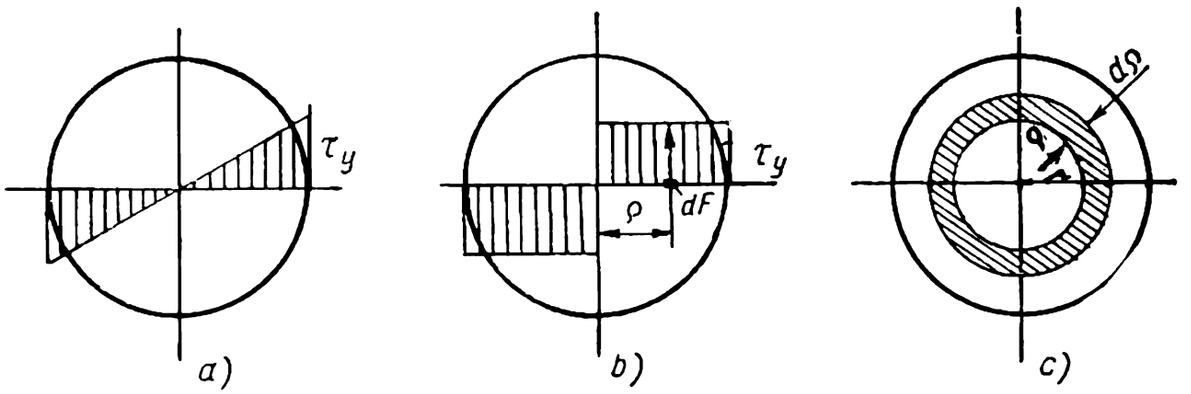


Fig. 13.4

mate twisting moment is greater than the dangerous moment based on elastic deformations alone. Their ratio is given by

$$\frac{M_{tu}}{M_{ty}}=\frac{W_{ppl}}{W_p}=\frac{\pi d^3}{12}\,\frac{16}{\pi d^3}=\frac{4}{3}. \tag{13.9}$$

Experiments show that the ultimate twisting moment as calculated from formula (13.7) is very close to experimental values.

Without giving the derivation we can write down the plastic section modulus in torsion for a ring cross section with outside diameter D and inside diameter d as

$$W_{ppl}=\frac{\pi D^3}{12}(1-c^3),$$

where $c=\frac{d}{D}$.

109. Plastic Bending of Statically Determinate Beams

In elastic bending the dangerous condition was the one in which the normal stresses at the extreme points of the cross section of a beam reached the yield point (Fig. 13.5*a*).

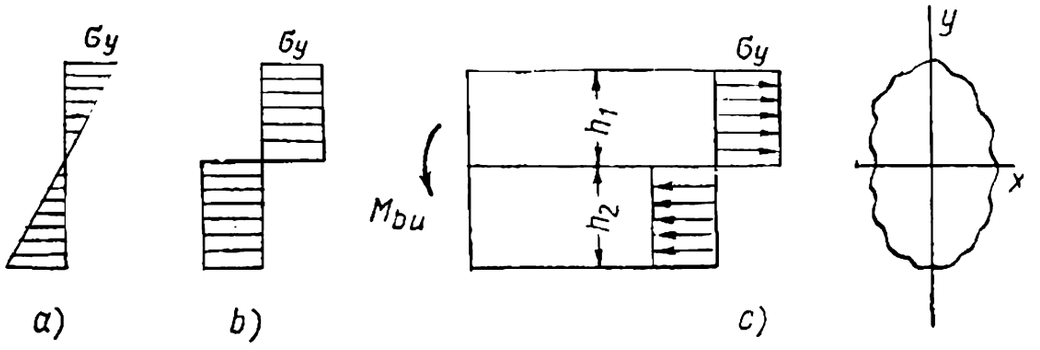


Fig. 13.5

The bending moment is then given by

$$M_{by} = W_x \sigma_y, \tag{13.10}$$

where W_x is the section modulus in bending; in the case of a rectangle, for instance,

$$W_x = \frac{bh^2}{6}.$$

However, as experiments show, such a condition is not limiting. A further increase in the load is possible owing to penetration of yielding into the internal fibres of the beam.

The stress diagram in the limiting condition is shown in Fig. 13.5*b*.

A further increase in the load is impossible (strain hardening of the material is neglected). At the dangerous cross section (where the maximum bending moment occurs) the so-called plastic hinge is formed, and the bending moment at this section reaches the ultimate value M_{bu}.

The magnitude of this moment will be determined from the condition that at all points of the dangerous cross section the normal stresses are equal to the yield point σ_y.

Consider first a case of pure plane bending. Determine the position of the neutral axis of the cross section. Noting

that the longitudinal force which represents the resultant of the internal normal forces $\sigma_y\, dF$ is equal to zero we obtain

$$N = \int_F \sigma_y\, dF = 0$$

or

$$\sigma_y \int_0^{h_1} b\, dy - \sigma_y \int_0^{h_2} b\, dy = 0,$$

where b is the width of the beam (variable).

The integrals represent the areas F_1 and F_2 of the stretched and compressed parts of the cross section. Thus

$$\sigma_y (F_1 - F_2) = 0, \tag{13.11}$$

whence

$$F_1 = F_2. \tag{13.12}$$

Consequently, *in plastic bending the neutral axis divides the cross section into two equivalent parts.*

Derive now the expression for the ultimate bending moment as the resultant moment of the internal normal forces $\sigma_y dF$:

$$M_{tu} = \int_F \sigma_y\, dFy$$

or

$$M_{bu} = \sigma_y \int_{F_1} y\, dF + \sigma_y \int_{F_2} y\, dF.$$

The integrals represent the static moments S_{1x} and S_{2x} of the upper and lower parts of the cross-sectional area with respect to the neutral axis. Therefore

$$M_{bu} = \sigma_y (S_{1x} + S_{2x}). \tag{13.13}$$

For sections having a horizontal axis of symmetry

$$S_{1x} = S_{2x} = S_{0x},$$

where S_{0x} is the static moment of the half-section with respect to the neutral axis and formula (13.13) becomes

$$M_{bu} = \sigma_y 2S_{0x} = \sigma_y W_{xpl}, \tag{13.14}$$

where $W_{xpl}=2S_{0x}$ is the so-called *plastic section modulus in bending*. For a rectangle, for instance,

$$W_{xpl}=b\frac{h}{2}\frac{h}{4}2=\frac{bh^2}{4}. \qquad (13.15)$$

Therefore, for a rectangular cross section

$$\frac{M_{bu}}{M_{by}}=\frac{W_{xpl}}{W_x}=\frac{bh^2}{4}:\frac{bh^2}{6}=1.5,$$

i.e., the ultimate bending moment is 50% greater in a design based on plastic deformations. For a circular cross section the ratio of the ultimate moments is equal to 1.7, for a ring cross section with the ratio $\frac{d}{D}=0.5$ the ratio of the moments is 1.57, for an I-section it is about 1.15.

The foregoing procedure, as experiments show, is suitable as a first approximation for the general case of bending as well, when both bending moment and shearing force occur at cross sections.

In the design of statically indeterminate beams consideration of plastic deformations enables one to discover still greater reserves of load-carrying capacity of a system by virtue of the equalisation of moments along the length of the beam.

110. Yield Conditions

Under combined stress it appears impossible to determine the ultimate values of internal force factors simply on the basis of equations of statics alone as in torsion or pure bending (i.e., in pure shear or uniaxial state of stress). For their determination, in addition to equations of equilibrium and continuity of deformations use is also made of the so-called yield conditions representing hypotheses which state the criteria for the onset of yielding of a material. At present the following two hypotheses are employed in practical calculations and theoretical investigations.

Hypothesis I (Saint-Venant's yield condition). Regardless of the state of stress yielding of the material occurs when the maximum shearing stress attains a certain definite value constant for a given material. This value of τ_{max} is determined experimentally in the uniaxial exten-

sion of a specimen. The mathematical expression of this yield condition leads to the criterion of equivalence according to the third theory of strength.

Hypothesis 2 (Huber-Mises yield condition). Regardless of the state of stress yielding of the material occurs when the potential distortion energy per unit of volume attains a certain definite value constant for a given material. This value of u_d is determined experimentally in the uniaxial extension of a specimen. The mathematical expression of this yield condition leads to the condition of equivalence according to the fourth theory of strength.

The second yield condition is in somewhat better agreement with experimental data than the first one, but the design relations are more involved.

111. Plastic Torsion with Tension of a Circular Bar

We shall use here the final result of the solution of this problem. *

It is assumed that the state of stress is the same for all cross sections of the bar. The longitudinal deformations are governed by the law of plane sections. It is also assumed that since deformations are small the contour of a cross section does not deform in its plane but rotates as a rigid disk.

With these hypotheses an equation is obtained relating the longitudinal force and twisting moment in the limiting state:

$$M_{tu}^2 + \frac{r^2 N_u^2}{3K_0} + \frac{N_u^3}{9\pi K_0 \sigma_y} = \frac{\sigma_y^2 W_{ppl}^2}{K_0}. \qquad (13.16)$$

Here r is the radius of the cross section of the bar, K_0 is a constant equal to 3 in cases where the first yield condition is used and $K_0=4$ if the second yield condition is employed, W_{ppl} is the plastic section modulus in torsion equal to

$$W_{ppl} = \frac{2}{3}\pi r^3.$$

---

* See Rzhanitsyn, A. R. *Structural Design Including Plastic Properties of Materials*. Moscow, Stroiizdat, 1949.

112. Plastic Bending with Tension and Shear

The results will be given of the approximate solution of this problem in a case of a rectangular cross section obtained by N. I. Bezukhov.

The equation governing the relation between the bending moment M_{bu}, the longitudinal force N_u and the shearing force Q_u in the limiting state is

$$M_{bu} + \frac{N_u^2}{4b\sigma_y} + \frac{9Q_u^2}{16b\sigma_y} = \sigma_y W_{xpl}, \qquad (13.17)$$

where $W_{xpl} = \frac{bh^2}{4}$ is the plastic section modulus in bending.

There is also a more exact solution of this problem. The design formulas are naturally more unwieldy.

113. The Basic Principles of Limit Design

The design method described in the preceding chapters is referred to as the elastic design (based on allowable stresses).

According to this method the strength and rigidity design of parts is carried out on the basis of Hooke's law. The dangerous condition of a part is assumed to be the one in which the stress at least at one (dangerous) point reaches the yield point.

Strength is considered to be assured in a part if the service (working) stress at the dangerous point does not exceed the allowable stress:

$$\sigma \leqslant [\sigma] = \frac{\sigma_y}{n}.$$

The safety factor is defined as the ratio of the yield point to the actual stress:

$$n = \frac{\sigma_y}{\sigma}.$$

However, as already mentioned, by using Hooke's law we are not in a position to calculate the ultimate loads the part can sustain just before fracture occurs or a certain amount of plastic deformation develops.

In many cases it is more logical to carry out the strength design under static loading on the basis of plastic

deformations, as shown in the present chapter, and to calculate the factor of safety as a ratio of the ultimate load Q_u to the working (actually applied) load Q:

$$n = \frac{Q_u}{Q}.$$

It is also more correct to use in the designing of parts a set of partial factors of safety, taking account of the effect of basic factors on the strength of a part rather than one factor of safety. This permits a more differentiated consideration of various concrete service conditions of a part.

For these reasons *the limit design of parts* which makes it possible to include plastic deformations in certain cases has become more widely used lately. In the design of engineering structures limit design was introduced as obligatory in 1955.

In mechanical engineering, owing to the greater variety of conditions, there is not such strict regulation of design methods.

At present in mechanical engineering the elastic design (based on allowable stresses) is still most widely used. However, the limit design is gaining in importance here as well.

The limiting states of a machine part can be taken as follows. *

1. The limiting state based on the load-carrying capacity which is characterised by the loads corresponding to the limiting states based on strength, stability, endurance, resistance to plastic deformations. These loads imply forces P, moments M, pressure q, etc., as the case may be.

2. The limiting state based on deformations and displacements, in which such deformations or vibrations occur in a part maintaining strength and stability as to make it unserviceable.

3. The limiting state based on the amount of wear of the surface layer of a part.

Each of these limiting states is characterised by the limiting condition that in order to secure safe service the forces acting on a part must be smaller than the ultimate value and in the limiting state equal to this value.

---

* Belyankin, F. P. *On Limit Design in Mechanical Engineering.* Kiev, Izd. Akademii nauk Ukr. S. S. R., 1953.

For the first limiting state this condition, called the condition of non-destruction, is

$$N \leqslant \Phi, \tag{13.18}$$

where N is the design force in a part, and Φ is the ultimate resistance of a part corresponding to the loss of load-carrying capacity. The design force in the part is determined as the sum of forces due to each normative load including the overloading factors n_1 greater than unity corresponding to each load.

Thus

$$N = \sum N_{\text{norm}} \times n_1. \tag{13.19}$$

The ultimate resistance of the part Φ depends on its shape and dimensions, resistance of the material to the corresponding type of deformation and service conditions of the part, i.e.,

$$\Phi = n_2 n_3 \sigma_{\text{norm}} \times F, \tag{13.20}$$

where n_3 is the factor of service conditions taking into account the degree of responsibility of the part and other characteristics which are not directly introduced into calculations; n_2 is the factor of homogeneity of the material; σ_{norm} is the normative resistance of the material (for ductile materials—the normative value of the yield point, for brittle materials—the normative value of the ultimate strength, in fatigue—the endurance limit); F is a geometrical factor of the part (area, section modulus, etc.).

In fatigue design the ultimate resistance should include the appropriate factors reducing the load-carrying capacity of the part (stress concentration factor, size factor, etc.).

Thus, in the expanded form the condition of non-destruction is

$$\sum N_{\text{norm}} n_1 = N \leqslant n_3 n_2 \sigma_{\text{norm}} \times F. \tag{13.21}$$

For the second limiting state the limiting condition is

$$\Delta \leqslant \Delta_u, \tag{13.22}$$

where Δ is the maximum displacement of the cross section of a part due to the working load, Δ_u is the ultimate displacement (depending on the function of the part) which makes it necessary to take the part out of service.

A more extensive use of the limit design method in mechanical engineering needs a more detailed standardisation of the factors of overloading, homogeneity, service conditions, ultimate amount of plastic deformation, ultimate wear, etc.

This will make it possible in selecting the dimensions of a part to rely not only on the experience and ingenuity of a designer but also on objective criteria and standards which will increase the reliability of parts and lead to a reduction in the use of materials in their production.

114. Conclusion. Current Trends in Development of Methods of Structural Design

The methods of structural design are continually developing and becoming more profound and perfect.

In recent years an extensive study has been made into the problems of dynamics of structures, the methods of structural design with proper account of plastic properties of materials, the methods of fatigue analysis, the problems associated with the use of computers in structural design based on strength and stability considerations and other problems.

Of particular note is a principally new method of structural design which has been lately worked out mainly in the U.S.S.R. (the work of N. S. Streletsky, A. R. Rzhanitsyn, V. V. Bolotin and others).

This method may conditionally be called the *statistical probability* (or simply *statistical*) *method.*

The principal idea of this method consists in the following. The quantities appearing in the equations of strength, stiffness and stability, viz., loads, material properties, geometrical characteristics of sections, are treated as random variables (statistical sets) with a certain, and sometimes rather an appreciable, variability (scatter) and not as constant strictly specified quantities. The study of such variables is possible only on the basis of the methods of probability theory.

A degree of scatter of one or another variable is characterised by the corresponding distribution curve.

By way of an example Fig. 1.6 represents the distribution curve for the wind load and Fig. 2.22 the distribution curve for the yield point of mild steel.

Even such supposedly constant quantities as the cross-sectional area, the section modulus, the moment of inertia and merely the linear dimensions of a part are actually statistically varying quantities due to inevitable errors involved in manufacture and measurements.

Distribution curves may be constructed from experimental data or in some cases from purely speculative considerations according to theoretical laws expressing the essential features of a given statistical set. There are several criteria available (Pearson, Kolmogorov, Bernstein) by means of which it is possible to establish that the theoretical curve is in fair agreement with the empirical one.

Distribution curves may be symmetrical (Fig. 2.22) and non-symmetrical (Fig. 1.6).

The most commonly used theoretical curve expressing a symmetrical distribution is the Gaussian curve, also called the *normal curve* (Fig. 2.22).

The Gaussian curve may also be employed for non-symmetrical distribution laws by "correcting" it in one way or another.

The preservation of the Gaussian law for describing non-symmetrical phenomena is desirable because the mathematical operations associated with this law are simplest, well studied and tabulated.

In selecting theoretical curves to fit experimental data use is generally made of the method of moments.

With the distribution curves of given variables available it is possible to construct the distribution curves of their functions by the rules of probability theory.

For instance, if the distribution curves of the load and the cross-sectional area are known, it is possible to construct the distribution curve of the normal stress $\sigma = \frac{P}{F}$ in central extension.

If the distribution curves of P and F are of normal form, the distribution curve of σ will also be normal. Then the condition of strength is written down which is expressed in terms of stresses as

$$\sigma < R, \tag{13.23}$$

where σ is the design stress, R is the limiting stress (the yield point for ductile materials).

When plastic deformations are taken into account stresses are replaced by forces

$$N < \Phi. \tag{13.24}$$

The question of the extent to which the design stress is to be reduced against the limiting stress is decided in the conventional procedure based on allowable stresses by introducing the factor of safety n.

In the statistical method it is accepted that the reduction of the quantities appearing in the right-hand members of (13.23) or (13.24) by a factor of n does not provide a hundred per cent guarantee against failure of a part since the distribution curves do not intersect the horizontal axis.

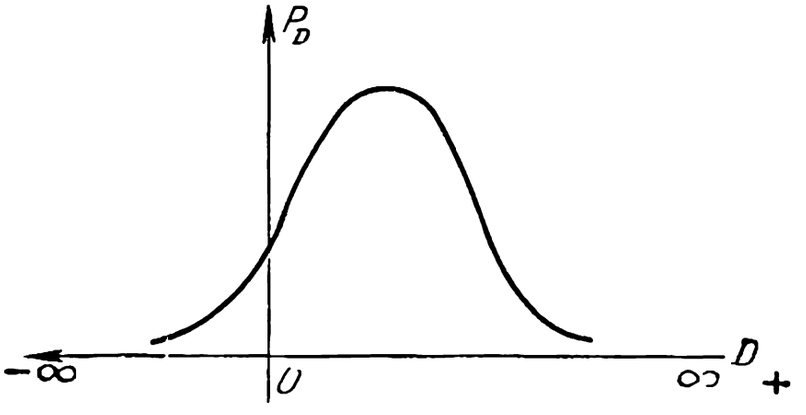


Fig. 13.6

Hence there always remains a certain probability of a part failing even at a large value of the factor of safety.

The problem is to know how to determine this probability of failure and to take one or another value as allowable depending on the purpose of a part. For that it is necessary to construct the distribution curve of the strength function $D = \Phi - N$ (or $R - \sigma$). When the distribution functions of R and σ are of normal form the curve of the function D will also be normal (Fig. 13.6). The vertical axis records the values of the density function of D, and the horizontal axis records the values of the function D itself. The total area under the curve is unity. The positive values of the function D correspond to safe cases of loading, the negative values correspond to cases of failure. The probability V of the failure of a part is characterised by the area under

the distribution curve of the function in the range from minus infinity to zero

$$V = \int_{-\infty}^{0} p_D \, dD. \tag{13.25}$$

Instead of the probability of failure it is possible to work with the probability of non-failure equal to the area under the positive portion of the distribution curve of the strength function

$$W = \int_{0}^{\infty} p_D \, dD = 1 - V. \tag{13.26}$$

There are tables available for the evaluation of integral (13.25). The calculated probability of the failure of a structure must not exceed a small quantity which may be termed the allowable probability of the failure of a structure.

The magnitude of the allowable probability of the failure of a part should be assigned according to the nature of consequences caused by the failure of the part.

If the failure of a part may entail human casualties, the probability of failure must be taken very small, practically equal to zero.

The investigations of Professor N. S. Streletsky showed that the probability of failure for completed structures ranges from 1×10^{-5} to 1×10^{-7}, i.e., one of a million structures may fail at any time on the average.

In the tentative standards for structural design it is recommended that the allowable probability of failure should be taken as 3×10^{-6} for especially responsible buildings and structures if their failure means a catastrophe.

If the failure of a part (structure) does not entail human casualties but involves only the necessity of repairing or replacing this part, the allowable probability of the failure of the part may be determined from pure economical considerations, namely from the condition that the total cost of the part (the cost of manufacture plus the cost of repair) is minimum.

The practical application of the statistical method must be preceded by extensive research work on the study of distribution curves of loads, material properties and other variables which govern the strength of a structure.

APPENDICES

APPENDIX I

GREEK ALPHABET

A, α	alpha	I, ι	iota	P, ρ	rho
B, β	beta	K, ϰ	kappa	Σ, σ, ς	sigma
Γ, γ	gamma	Λ, λ	lambda	T, τ	tau
Δ, δ	delta	M, μ	mu	Υ, υ	upsilon
E, ε	epsilon	N, ν	nu	Φ, φ	phi
Z, ζ	zeta	Ξ, ξ	xi	X, χ	chi
H, η	eta	O, o	omicron	Ψ, ψ	psi
Θ, θ	theta	Π, π	pi	Ω, ω	omega

APPENDIX II

Symbols of Divisions and Functions of Units

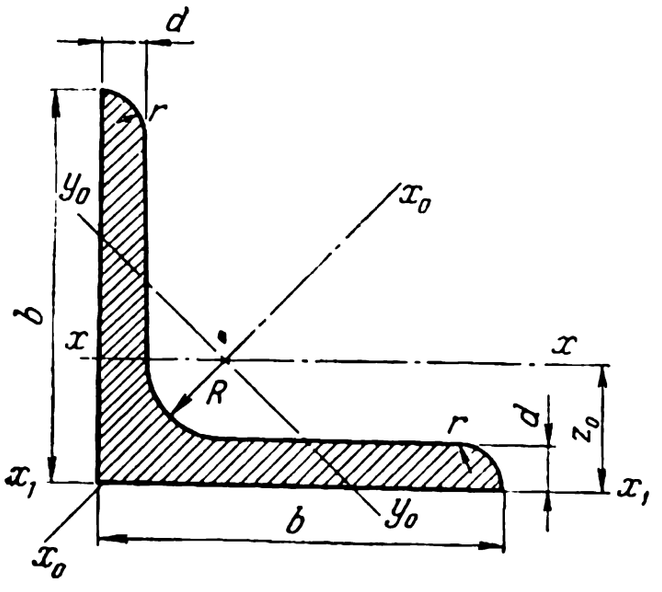
Prefix to primary unit	Magnitude ratio	Symbols		Prefix to primary unit	Magnitude ratio	Symbols	
		In Russian	In Latin or Greek			In Russian	In Latin or Greek
Pico	10^{-12}	*п*	p	Deca	10	*да*	da
Nano	10^{-9}	*н*	n	Hecto	10^2	*г*	h
Micro	10^{-6}	*мк*	μ	Kilo	10^3	*к*	k
Milli	10^{-3}	*м*	m	Mega	10^6	*М*	M
Centi	10^{-2}	*с*	c	Giga	10^9	*Г*	G
Deci	10^{-1}	*д*	d	Tera	10^{12}	*Т*	T

APPENDIX III

Mechanical Quantities and Units of SI (mks) System

Quantity	Dimension	Symbol		Name
		Russian	Internat.	
Length	m	*м*	m	metre
Mass	kg	*кг*	kg	kilogram
Time	sec	*сек*	sec	second
Area	m^2	—	—	square metre
Volume	m^3	—	—	cubic metre
Plane angle	—	*рад*	rad	radian
Solid angle	—	*стер*	sr	steradian
Linear velocity	$m\ sec^{-1}$	—	—	—
Angular velocity	sec^{-1}	—	—	—
Linear acceleration	$m\ sec^{-2}$	—	—	—
Angular acceleration	sec^{-2}	—	—	—
Force	$m\ kg\ sec^{-2}$	*н*	N	newton
Weight	$m\ kg\ sec^{-2}$	*н*	N	newton
Specific weight	$m^{-2}\ kg\ sec^{-2}$	—	—	—
Density	$m^{-3}\ kg$	—	—	—
Impulse of force	$m\ kg\ sec^{-1}$	—	—	—
Momentum	$m\ kg\ sec^{-1}$	—	—	—
Moment of force	$m^2\ kg\ sec^{-2}$	—	—	—
Pressure	$m^{-1}\ kg\ sec^{-2}$	—	—	—
Work	$m^2\ kg\ sec^{-2}$	*дж*	j	joule
Energy	$m^2\ kg\ sec^{-2}$	*дж*	j	joule
Power	$m^2\ kg\ sec^{-3}$	*вт*	w	watt
Moment of inertia	$m^2\ kg$	—	—	—
Coefficient of longitudinal extension	$m\ kg^{-1}\ sec^2$	—	—	—
Coefficient of lateral contraction	$m\ kg^{-1}\ sec^2$	—	—	—
Young's modulus	$m^{-1}\ kg\ sec^{-2}$	—	—	—
Shear modulus	$m^{-1}\ kg\ sec^{-2}$	—	—	—
Period of vibration	sec	*T*	—	—
Frequency of vibration	sec^{-1}	*гц*	hz	hertz
Angular frequency	sec^{-1}	—	—	—

ROLLED STEEL SECTIONS



Rolled steel
Equal angles
GOST 8509-57

No. of section	Dimensions				Area of sect.	Weight per metre	Data for axes								
							x-x		x_0-x_0		y_0-y_0		x_1-x_1	z_0
	b	a	R	r			J_x	i_x	J_{x_0} max.	i_{x_1} max.	J_{y_0} min.	i_{y_0} min.	J_{x_1}	
	mm				cm^2	kg	cm^4	cm	cm^4	cm	cm^4	cm	cm^4	cm
2	20	3	3.5	1.2	1.13	0.89	0.40	0,59	0.63	0.75	0,17	0.39	0.81	0.60
		4			1.46	1.15	0.50	0,58	0.78	0.73	0,22	0.38	1.09	0.64

2.5	25	3 4	3.5	1.2	1.43 1.86	1.12 1.46	0.81 1.03	0.75 0.74	1.29 1.62	0.95 0.93	0.34 0.44	0.49 0.48	1.57 2.11	0.73 0.76
2.8	28	3	4	1.3	1.62	1.27	1.16	0.85	1.84	1.07	0.48	0.55	2.20	0.80
3.2	32	3 4	4.5	1.5	1.86 2.43	1.46 1.91	1.77 2.26	0.97 0.96	2.8 3.58	1.23 1.21	0.74 0.94	0.63 0.62	3.26 4.39	0.89 0.94
3.6	36	3 4	4.5	1.5	2.10 2.75	1.65 2.16	2.56 3.29	1.10 1.09	4.06 5.21	1.39 1.38	1.06 1.36	0.71 0.70	4.64 6.24	0.99 1.04
4	40	3 4	5	1.7	2.35 3.08	1.85 2.42	3.55 4.58	1.23 1.22	5.63 7.26	1.55 1.53	1.47 1.90	0.79 0.78	6.35 8.53	1.09 1.13
4.5	45	3 4 5	5	1.7	2.65 3.48 4.29	2.08 2.73 3.37	5.13 6.63 8.03	1.39 1.38 1.37	8.13 10.5 12.7	1.75 1.74 1.72	2.12 2.74 3.33	0.89 0.89 0.88	9.04 12.1 15.3	1.21 1.26 1.30
5.0	50	3 4 5	5.5	1.8	2.96 3.89 4.80	2.32 3.05 3.77	7.11 9.21 11.2	1.55 1.54 1.53	11.3 14.6 17.8	1.95 1.94 1.92	2.95 3.80 4.63	1.00 0.99 0.98	12.4 16.6 20.9	1.33 1.38 1.42
5.6	56	3.5 4 5	6	2	3.86 4.38 5.41	3.03 3.44 4.25	11.6 13.1 16.0	1.73 1.73 1.72	18.4 20.8 25.4	2.18 2.18 2.16	4.80 5.41 6.59	1.12 1.11 1.10	20.3 23.3 29.2	1.50 1.52 1.57
6.3	63	4 5 6	7	2.3	4.96 6.13 7.28	3.90 4.81 5.72	18.9 23.1 27.1	1.95 1.94 1.93	29.9 36.6 42.9	2.45 2.44 2.43	7.81 9.52 11.2	1.25 1.25 1.24	33.1 41.5 50.0	1.69 1.74 1.78

Appendix IV (continued)

No. of section	Dimensions				Area of sect.	Weight per metre	Data for axes							
							x-x		x_0-x_0		y_0-y_0		x_1-x_1	z_0
	b	a	R	r			J_x	i_x	J_{x_0} max.	i_{x_0} max.	J_{y_0} min.	i_{y_0} min.	J_{x_1}	
	mm				cm^2	kg	cm^4	cm	cm^4	cm	cm^4	cm	cm^4	cm
7	70	4.5	8	2.7	6.2	4.87	29.0	2.16	46.0	2.72	12.0	1.39	51.0	1.88
		5			6.86	5.38	31.9	2.16	50.7	2.72	13.2	1.39	56.7	1.90
		6			8.15	6.39	37.6	2.15	59.6	2.71	15.5	1.38	68.4	1.94
		7			9.42	7.39	43.0	2.14	68.2	2.69	17.8	1.37	80.1	1.99
		8			10.7	8.37	48.2	2.13	76.4	2.68	20.0	1.37	91.9	2.02
7.5	75	5	9	3	7.39	5.8	39.5	2.31	62.6	2.91	16.4	1.49	69.6	2.02
		6			8.78	6.89	46.6	2.30	73.9	2.90	19.3	1.48	83.9	2.06
		7			10.1	7.96	53.3	2.29	84.6	2.89	22.1	1.48	98.3	2.10
		8			11.5	9.02	59.8	2.28	94.9	2.87	24.8	1.47	113	2.15
		9			12.8	10.1	66.1	2.27	105	2.86	27.5	1.46	127	2.18
8	80	5.5	9	3	8.63	6.78	52.7	2.47	83.6	3.11	21.8	1.59	93.2	2.17
		6			9.38	7.36	57.0	2.47	90.4	3.11	23.5	1.58	102	2.19
		7			10.8	8.51	65.3	2.45	104	3.09	27.0	1.58	119	2.23
		8			12.3	9.65	73.4	2.44	116	3.08	30.3	1.57	137	2.27

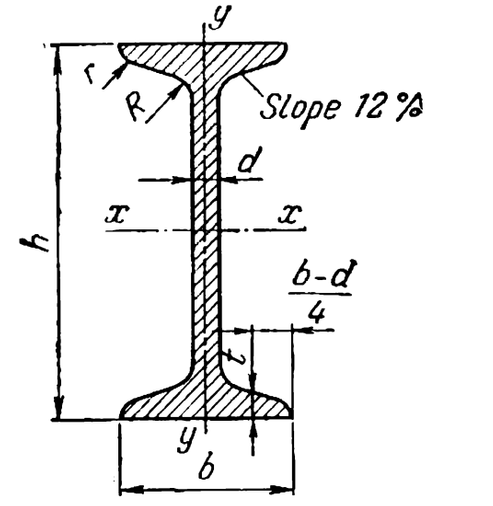
9	90	6	10	3.3	10.6	8.33	82.1	2.78	130	3.50	34.0	1.79	145	2.43
		7			12.3	9.64	94.3	2.77	150	3.49	38.9	1.78	169	2.47
		8			13.9	10.9	106	2.76	168	3.48	43.8	1.77	194	2.51
		9			15.6	12.2	118	2.75	186	3.46	48.6	1.77	219	2.55
10	100	6.5	12	4	12.8	10.1	122	3.09	193	3.88	50.7	1.99	214	2.68
		7.0			13.8	10.8	131	3.08	207	3.88	54.2	1.98	231	2.71
		8			15.6	12.2	147	3.07	233	3.87	60.9	1.98	265	2.75
		10			19.2	15.1	179	3.05	284	3.84	74.1	1.96	333	2.83
		12			22.8	17.9	209	3.03	331	3.81	86.9	1.95	402	2.91
		14			26.3	20.6	237	3.00	375	3.78	99.3	1.94	472	2.99
		16			29.7	23.3	264	2.98	416	3.74	112	1.94	542	3.06
11	110	7	12	4	15.2	11.9	176	3.40	279	4.29	72.7	2.19	308	2.96
		8			17.2	13.5	198	3.39	315	4.28	81.8	2.18	353	3.00
12,5	125	8	14	4.6	19.7	15.5	294	3.87	467	4.87	122	2.49	516	3.36
		9			22.0	17.3	327	3.86	520	4.86	135	2.48	582	3.40
		10			24.3	19.1	360	3.85	571	4.84	149	2.47	649	3.45
		12			28.9	22.7	422	3.82	670	4.82	174	2.46	782	3.53
		14			33.4	26.2	482	3.80	764	4.78	200	2.45	916	3.61
		16			37.8	29.6	539	3.78	853	4.75	224	2.44	1051	3.68
14	140	9	14	4.6	24.7	19.4	466	4.34	739	5.47	192	2.79	818	3.78
		10			27.3	21.5	512	4.33	814	5.46	211	2.78	911	3.82
		12			32.5	25.5	602	4.31	957	5.43	248	2.76	1097	3.90

Appendix IV (continued)

No. of section	Dimensions				Area of sect.	Weight per metre	Data for axes							
							x-x		x_0-x_0		y_0-y_0		x_1-x_1	z_0
	b	a	R	r			J_x	i_x	J_{x_0} max.	i_{x_0} max.	J_{y_0} min.	i_{y_0} min.	J_{x_1}	
	mm				cm^2	kg	cm^4	cm	cm^4	cm	cm^4	cm	cm^4	cm
16	160	10	16	5.3	31.4	24.7	774	4.96	1229	6.25	319	3.19	1356	4.30
		11			34.4	27.0	844	4.95	1341	6.24	348	3.18	1494	4.35
		12			37.4	29.4	913	4.94	1450	6.23	376	3.17	1633	4.39
		14			43.3	34.0	1046	4.92	1662	6.20	431	3.16	1911	4.47
		16			49.1	38.5	1175	4.89	1866	6.17	485	3.14	2191	4.55
		18			54.8	43.0	1299	4.87	2061	6.13	537	3.13	2472	4.63
		20			60.4	47.4	1419	4.85	2248	6.10	589	3.12	2756	4.70
18	180	11	16	5.3	38.8	30.5	1216	5.60	1933	7.06	500	3.59	2128	4.85
		12			42.2	33.1	1317	5.59	2093	7.04	540	3.58	2324	4.89

20	200	12	18	6	47.1	37.0	1823	6.22	2896	7.84	749	3.99	3182	5.37
		13			50.9	39.9	1961	6.21	3116	7.83	805	3.98	3452	5.42
		14			54.6	42.8	2097	6.20	3333	7.81	861	3.97	3722	5.46
		16			62.0	48.7	2363	6.17	3755	7.78	970	3.96	4264	5.54
		20			76.5	60.1	2871	6.12	4560	7.72	1182	3.93	5355	5.70
		25			94.3	74.0	3466	6.06	5494	7.63	1432	3.91	6733	5.89
		30			111.5	87.6	4020	6.00	6351	7.55	1688	3.89	8130	6.07
22	220	14	21	7	60.4	47.4	2814	6.83	4470	8.60	1159	4.38	4941	5.93
		16			68.6	53.8	3175	6.81	5045	8.58	1306	4.36	5661	6.02
25	250	16	24	8	78.4	61.5	4717	7.76	7492	9.78	1942	4.98	8286	6.75
		18			87.7	68.9	5247	7.73	8337	9.75	2158	4.96	9342	6.83
		20			97.0	76.1	5765	7.71	9160	9.72	2370	4.94	10401	6.91
		22			106.1	83.3	6270	7.69	9961	9.69	2579	4.93	11464	7.00
		25			119.7	94.0	7006	7.65	11125	9.64	2887	4.91	13064	7.11
		28			133.1	104.5	7717	7.61	12244	9.59	3190	4.89	14674	7.23
		30			142.0	111.4	8117	7.59	12965	9.56	3389	4.89	15573	7.31

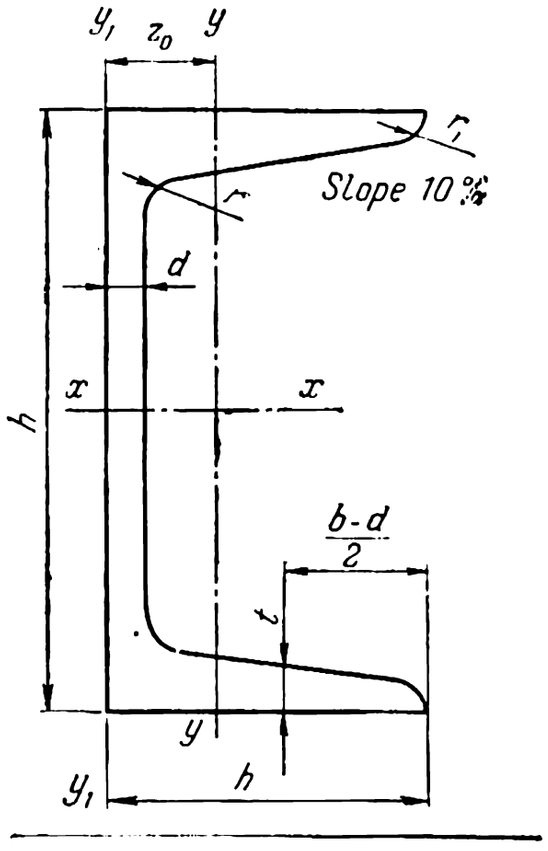
Appendix IV (continued)



Rolled steel
I-beams
GOST 8239-56

No. of section	Weight per metre, kg	Dimensions						Area of sect., cm²	Data for axes						
									x-x				y-y		
		h	b	d	t	R	r		J_x	W_x	i_x	S_x	J_y	W_y	i_y
		mm							cm⁴	cm³	cm	cm³	cm⁴	cm³	cm
10	11.1	100	70	4.5	7.2	7.0	3.0	14.2	244	48.8	4.15	28.0	35.3	10.1	1.58
12	13.0	120	75	5.0	7.3	7.5	3.0	16.5	403	67.2	4.94	38.5	43.8	11.7	1.63
14	14.8	140	82	5.0	7.5	8.0	3.0	18.9	632	90.3	5.78	51.5	58.2	14.2	1.75
16	16.9	160	90	5.0	7.7	8.5	3.5	21.5	945	118	6.63	67.0	77.6	17.2	1.90
18	18.7	180	95	5.0	8.0	9.0	3.5	23.8	1330	148	7.47	83.7	94.6	19.9	1.99

18a	19.9	180	102	5.0	8.2	9.0	3.5	25.4	1440	160	7.53	90.1	119	23.3	2.17
20	20.7	200	100	5.2	8.2	9.5	4.0	26.4	1810	181	8.27	102	112	22.4	2.06
20a	22.2	200	110	5.2	8.3	9.5	4.0	28.3	1970	197	8.36	111	148	27.0	2.29
22	23.7	220	110	5.3	8.6	10.0	4.0	30.2	2530	230	9.14	130	155	28.2	2.26
22a	25.4	220	120	5.3	8.8	10.0	4.0	32.4	2760	251	9.23	141	203	33.8	2.50
24	27.3	240	115	5.6	9.5	10.5	4.0	34.8	3460	289	9.97	163	198	34.5	2.37
24a	29.4	240	125	5.6	9.8	10.5	4.0	37.5	3800	317	10.1	178	260	41.6	2.63
27	31.5	270	125	6.0	9.8	11.0	4.5	40.2	5010	371	11.2	210	260	41.5	2.54
27a	33.9	270	135	6.0	10.2	11.0	4.5	43.2	5500	407	11.3	229	337	50.0	2.80
30	36.5	300	135	6.5	10.2	12.0	5.0	46.5	7080	472	12.3	268	337	49.9	2.69
30a	39.2	300	145	6.5	10.7	12.0	5.0	49.9	7780	518	12.5	292	436	60.1	2.95
33	42.2	330	140	7.0	11.2	13.0	5.0	53.8	9840	597	13.5	339	419	59.9	2.79
36	48.6	360	145	7.5	12.3	14.0	6.0	61.9	13380	743	14.7	423	516	71.1	2.89
40	56.1	400	155	8.0	13.0	15.0	6.0	71.4	18930	947	16.3	540	666	75.9	3.05
45	65.2	450	160	8.6	14.2	16.0	7.0	83.0	27450	1220	18.2	699	807	101	3.12
50	76.1	500	170	9.3	15.2	17.0	7.0	96.9	39120	1560	20.1	899	1040	122	3.28
55	88.6	550	180	10.0	16.5	18.0	7.0	113	54810	1990	22.0	1150	1350	150	3.46
60	103	600	190	11.8	17.8	20.0	8.0	131	75010	2500	23.9	1440	1720	181	3.62
65	119	650	200	11.7	19.2	22.0	9.0	151	100840	3100	25.8	1790	2170	217	3.79
70	137	700	210	12.7	20.8	24.0	10.0	174	133890	3830	27.7	2220	2730	260	3.96
70a	158	700	210	15.0	24.0	24.0	10.0	202	152700	4360	27.5	2550	3240	309	4.01
706	184	700	210	17.5	28.2	24.0	10.0	234	175350	5010	27.4	2940	3910	373	4.09



Rolled steel
Channels
GOST 8240-56

No. of section	Weight per metre, kg	Dimensions						Area of sect., cm²	Data for axes							
									x-x				y-y			z_0
		h	b	d	t	r	r_1		J_x	W_x	i_x	S_x	J_y	W_y	i_y	
		mm							cm⁴	cm³	cm	cm³	cm⁴	cm³	cm	cm
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
5	5.42	50	37	4.5	7.0	6.0	2.5	6.90	26.1	10.4	1.94	6.36	8.41	3.59	1.10	1.36

6.5	6.50	65	40	4.5	7.4	6.0	2.5	8.28	54.5	16.8	2.57	10.0	11.9	4.58	1.20	1.40
8	7.78	80	45	4.8	7.4	6.5	2.5	9.91	99.9	25.0	3.17	14.8	17.8	5.89	1.34	1.48
10	9.20	100	50	4.8	7.5	7.0	3.0	11.7	187	37.3	3.99	21.9	25.6	7.42	1.48	1.55
12	10.8	120	54	5.0	7.7	7.5	3.0	13.7	313	52.2	4.78	30.5	34.4	9.01	1.58	1.59
14	12.3	140	58	5.0	8.0	8.0	3.0	15.7	489	69.8	5.59	40.7	45.1	10.9	1.70	1.66
14a	13.2	140	62	5.0	8.5	8.0	3.0	16.9	538	76.8	5.65	44.6	56.6	13.0	1.83	1.84
16	14.1	160	64	5.0	8.3	8.5	3.5	18.0	741	92.6	6.42	53.7	62.6	13.6	1.87	1.79
16a	15.1	160	68	5.0	8.8	8.5	3.5	19.3	811	101	6.48	58.5	77.3	16.0	2.00	1.98
18	16.1	180	70	5.0	8.7	9.0	3.5	20.5	1080	120	7.26	69.4	85.6	16.9	2.04	1.95
18a	17.2	180	74	5.0	9.2	9.0	3.5	21.9	1180	131	7.33	75.2	104	19.7	2.18	2.13
20	18.4	200	76	5.2	9.0	9.5	4.0	23.4	1520	152	8.07	87.8	113	20.5	2.20	2.07
20a	19.6	200	80	5.2	9.6	9.5	4.0	25.0	1660	166	8.15	95.2	137	24.0	2.34	2.27
22	20.9	220	82	5.3	9.6	10.0	4.0	26.7	2120	193	8.91	111	151	25.4	2.38	2.24
22a	22.5	220	87	5.3	10.2	10.0	4.0	28.6	2320	211	9.01	121	186	29.9	2.55	2.47
24	24.0	240	90	5.6	10.0	10.5	4.0	30.6	2900	242	9.73	139	208	31.6	2.60	2.42
24a	25.8	240	95	5.6	10.7	10.5	4.0	32.9	3180	265	9.84	151	254	37.2	2.78	2.67
27	27.7	270	95	6.0	10.5	11	4.5	35.2	4160	308	10.9	178	262	37.3	2.73	2.47
30	31.8	300	100	6.5	11.0	12	5.0	40.5	5810	387	12.0	224	327	43.6	2.84	2.52
33	36.5	330	105	7.0	11.7	13	5.0	46.5	7980	484	13.1	281	410	51.8	2.97	2.59
36	41.9	360	110	7.5	12.6	14	6.0	53.4	10820	601	14.2	350	513	61.7	3.10	2.68
40	48.3	400	115	8.0	13.5	15	6.0	61.5	15220	761	15.7	444	642	73.4	3.23	2.75

SUBJECT INDEX

Allowable stress, 61, 137
Angle of twist, 140
  signs for, 143
Angles, section properties of, 386-391
Area, centroid of, 113
  moment of, 111
  moment of inertia of, 113, 116
  principal axes for, 121
  product of inertia of, 114, 124
  radius of gyration of, 289
Areas, principal, 28, 84
Axially loaded members, 36-41
  statically indeterminate, 68
    inelastic behaviour of, 368
Axis, centroidal, 112
  neutral, 179
  principal, 121

Beams, 160
  broken, 176, 229
  cantilever, 162, 207, 224
    shear and bending-moment diagrams for, 168, 170, 173
    deflection curve for, 202
  continuous, 233
  deflection of, 199
  economical sections of, 250
  elastic curve equation of, 200
  fixed at one end, supported at other, 234
  I, 191
  inelastically strained, 373
  maximum shearing stress in rectangular, 190
  multispan, 233, 248
  neutral axis of, 179
  normal stresses in, 183
  on three supports, 236
  overhanging, 174, 210, 226
  radius of curvature, 182
  rational arrangement of supports of, 246
  shearing stresses in, 189
  simple, 208, 209
    shear and bending-moment diagrams for, 172
  statically indeterminate, 162, 231
  slope of, 199
Bending, curvature, 182
  definition of, 25, 160
  energy of, 197
  induced, 324
  inelastic, 373
  oblique, 160, 277
  of thin-walled bars, 193
  plane, 160
  pure, 160
  sign conventions for, 165
  transverse, 160
Bending moment, 24
  algebraic expressions for, 165
  fully plastic value of, 374
  relation to shear, 166
  signs for, 165
Bending-moment diagram, 168
Biaxial stress, 29, 82, 87
Brittleness, 47
Buckling, of columns, 311
  critical load for, 312
  effect of end conditions on, 315
  empirical formulas for, 318
  Euler's formula for, 313
  factor of safety for, 312
  stress, 316

Centre of shear, 194
Centroid of area, 113
Channels, section properties of, 394-395
Chernov's lines, 45
Coefficient of thermal expansion, 73
Columns, buckling of, 311
  critical load, 312
  critical stress, 316
  economical sections of, 321
  effect of end conditions, 315
  Euler's formula, 313
  safety factor, 312
  with high slenderness ratios, 317
Combined loading, 26
  axial and bending, 285
  axial and torsional, 300
  bending and torsion, 297, 303
  shear and torsion, 293
Compatibility, equations of, 68
Compression, definition of, 25
Compression members, 311
  economical sections of, 321
Compression test, 54
Compressive strength, ultimate, 55
Contact stresses, 96
Continuity, 13
  equations of, 68
Continuous beams, 233
  three-moment equation for, 234
Contraction, definition of, 22
Core of cross section, 292
Creep, 51
Creep limit, definition of, 51
Critical load, columns, 312
Crushing stresses, 96, 104
Curvature, radius of, 182
Cycle of stress, 347

Deflection, of broken beam, 229
  by Mohr's method, 216
  of cantilever beam, 202, 207, 224
  of overhanging beam 210, 226
  of simple beam, 208, 209
  of statically indeterminate beams, 231
Deflection formula for beams, universal, 206
Deformation, defined, 22
  (*see also* Strain)
Design, defined, 9-10
  limit, 377
  methods of, 380
Displacements, 22
Distributed force, 19
Distribution, frequency, 20, 58, 381
  of loads in joints, 101
Distribution curves, 20, 58, 381
Ductility, 47
Dynamic loads, 19, 330

Eccentric axial loading, 287
Effective stress concentration factor, 96
Elastic after-effects, 50
Elastic curve of beam, equation of, 200
Elastic limit, 43
Elasticity, 23
  modulus of, 35
Elongation, definition of, 22
  ultimate, 45
Endurance limit, 350
Energy, elastic strain, 92
  in bending, 197
  in direct stress, 92
  in shear, 107
  in torsion, 144
  per unit volume, 92
Equilibrium, stable, 10, 311
  unstable, 10, 311
Euler's formula, 313
  effect of end conditions, 315
  limitations of, 316

Factor of safety, 61
  for buckling, 312
  for combined loading, 302
  for cyclic loads, 359, 360
Failure, probability of, 382
  strength theories of, 257
Fatigue, 345
  effect on, of hardening, 359
    of size, 357
    of stress concentration, 355
    of surface condition, 359
  endurance limit, 350
  factor of safety for, 359, 360
  mechanism of, 345

reduction of harmful effects of, 363
S-N curves, 350
stress concentration factor for, 355
testing machine, 349
under combined loading, 363
Fatigue crack, 345
Flexural rigidity, 183
Flexure formula, 183
Forces, distributed, 19
external, 18
general principles involved in computation, 25
internal, 15, 164
longitudinal, 24
transverse, 24
Force method for finding statically indeterminate forces and moments, 238
Frames, method of analysis of, 238
Frequency-distribution curves, 20, 58, 381

Gauss law of frequency distribution, 381

Homogeneity, 14
Hooke's law, 35, 106
generalised, 87
Huber-Mises yield condition, 376
Hypothesis of plane sections, 33, 178

I-beams, eccentrically loaded in compression, 287
section properties of, 392-393
Impact, 333
Impact test, 336
Induced bending, 324
Inelastic behaviour, 366
bending, 373
bending, tension, and shear, 377
torsion, 371
torsion and tension, 376
Inelastic deformation, 366
Inertia, moment of, 113, 116
product of, 114, 122
Instability, 10, 311
(*see also* Buckling)
Isotropy, 14

Joints, load distribution in, 101
riveted, 100, 108
welded, 105, 110

Limit design, 377
Loads, concentrated, 18
distributed, 19
dynamic, 19
eccentric, 287
impact, 19
repeated, 19
static, 19

Materials, properties of, 41
Maximum-energy theory of strength, 268, 270
Maximum-normal-stress theory of strength, 263
Maximum-shear-stress theory of strength, 266
Maximum-strain theory of strength, 265
Maxwell's reciprocal theorem, 216
Method of sections, 23
Modulus, of elasticity, 35
of rigidity, 106
section, 137, 184
Mohr's method for finding deflections, 216
application to broken beam, 229
application to cantilever beam, 224
application to overhanging beam, 226
Mohr's theory of strength, 270
Moment, bending (*see* Bending moment)
of inertia, 113, 116
centroidal, 116
composite areas, 118
parallel-axis theorem, 115
principal, 121
principal axes of, 121
polar, 113
static, 111

Necking, 45
Neutral axis, 179
Non-homogeneity, 58

Normal curve, 381
Normal stress, 27
  maximum, 81
  sign of, 27

Parallel-axis theorem, 115
Percentage, of reduction of area, 46
  of ultimate elongation, 45
Permanent strain or set, 23
Plastic deformation, 366
  (*see also* Inelastic deformation)
Plastic hinge, 373
Plasticity, 23
Plastics, properties of, 55
Poisson's ratio, 36
  limiting values of, 89
Polar moment of inertia, 113
Principal axes, 121
Principal moment of inertia, 121
Principal strains, 89
Principal stresses, 28, 84
Principle, of superposition, 15
  Saint-Venant's, 15
Probability of failure, 382
Product of inertia, 114, 124
  parallel-axis theorem for, 122
Properties, of an area, 111
  of materials, 41
    scatter in, 58
  of rolled-steel sections, tables of, 386-395
Proportional limit, 43
Pure bending, 160
Pure shear, 84, 139

Radius, of curvature of a beam, 182
  of gyration, 289
Reactions, 162
Reciprocal theorem, 212
Rectangular cross section, torsion of, 146
Reduction of area, 46
Relaxation, 52
Reloading, effects of, 49
Resonance, 342
Rigidity, flexural, 183
  modulus of, 106
  torsional, 140
Riveted joints, design of, 100
  butt joints, 108
  double shear in, 100
  lap joints, 100
  single shear, 100

S-N diagram, 350
  method of plotting, 351
Safety, factor of, 61
  for buckling, 312
  for combined loading, 302
  for cyclic loads, 359, 360
Saint-Venant's principle, 15
Saint-Venant's yield condition, 375
Scatter in properties of materials, 58
Section modulus, in bending, 184
  in torsion, 137
Separated moment diagrams, 221
Shear, 100
  centre of, 194
  definition of, 25
  pure, 84, 139
Shear centre, 194
Shear modulus of elasticity, 106
Shearing force, 24
  algebraic expressions for, 165
  relation to moment, 166
  signs for, 165
Shearing-force diagram, 168
Shearing strain, 22, 106
Shearing stress, 27
  in axial loading, 79
  in torsion, 135
  maximum, 81, 83
    in terms of principal stresses, 81, 83
  sign of, 80
Slenderness ratio, 317
Slopes of beams, 199
Specimen, compression test, 54
  impact test, 336
  tensile test, 42
Springs, close-coiled, design of, 293
State of stress, 28
Static loads, 19
Static moment of area, 111
Statically determinate beams, 162
Statically indeterminate members, 68, 162, 231
  beams fixed at one end, supported at other, 234

beams on three supports, 236
fully plastic load for, 370
inelastic behaviour of, 368
method of analysis of, 238
torsional members, 152
Statistical aspects, 19, 58
Statistical methods, 19, 61, 380
Stiffness, 10
Strain, elastic, 23
lateral, 36
plastic, 23
principal, 89
shearing, 22, 106
tensile or compressive, 23
total, 22
true, 46
volumetric, 89
Strain energy, elastic, 92
in bending, 197
in direct stress, 92
in shear, 107
in torsion, 144
per unit volume, 92
Strain hardening, 50
Strength theories, 257
Stress, allowable, 61, 137
concentration, 94, 155, 195, 355
concentration factor, bending loads, 196
direct central loads, 95, 96
repeated loads, 355
torsional loads, 155
contact, 96
critical, 316
cycle of, 347
definition of, 26
due to dynamic load, 331
initial, 73
maximum normal, 81
maximum shearing, 81
normal, 27
on oblique section, 79, 82
plane state of, 29
principal, 28, 84
reversal of, 349
shearing, 27
thermal, 73
total, 27
true, 46
uniform, 84
units of, 17
yield, 44
Stress concentration, 94
effect on fatigue, 355
in bending, 195
in torsion, 155
Stress concentration factor, at fillets in torsion, 155
effective values of, 96
in bending, 196
in fatigue, 355
theoretical values for, 95
Stress rupture, 53
Stress-strain curves, flat top, 46
with no yield point, 46
with yield point, 43
Stress-strain relationships, for biaxial stress, 88
for inelastic behaviour, 366
for triaxial stress, 89
for uniaxial stress, 35
Superposition, principle of, 15
Systems of mechanical units, 16, 384-385

Temperature, effects of, 52, 338
Tensile strength, effect of alloy additions on, 53
effect of size on, 48
effect of temperature on, 52
ultimate, 45
Tensile test, 42
diagram of, 43
true stress-strain in, 46
Tension, definition of, 25
Test bars, compression, 54
impact, 336
tensile, 42
Theoretical stress concentration factor, 95
Theories of strength, 257
combined theories, 271
energy-of-distortion, 269
maximum energy, 268, 270
maximum normal stress, 263
maximum shear stress, 266
maximum strain, 265
Mohr, 270
Thermal stresses, 73
Thin-walled bars, bending of, 193
torsion of, 149
Thin-walled vessels, 305
Three-moment equation for continuous beams, 234
Time effects, 50
Torsion, analysis of, 130
angle of twist of, 140

axial stresses caused by constraint, 148
definition of, 25
economical sections, 156
energy of, 144
inelastic, 371
of non-circular bars, 146
of open sections, 152
of rectangular cross section, 146
of round bars, 133, 140
of thin-walled bars, 149
section modulus, 137
shearing stresses in, 135
with one section restrained from warping, 148
warping of cross section in, 146
Torsion formula, 137
Torsional rigidity, 140
Toughness, 338
effect of temperature on, 339
Triaxial stress, 28, 87
True strain, 46
True stress, 46
Twisting moment, 25, 131
fully plastic value of, 371
in terms of horsepower, 139
signs for, 132
Twisting-moment diagram, 130

Ultimate compressive strength, 55
Ultimate tensile strength, 45
Uniaxial stress, 29, 79

Vibration, forced, 342
free, 337
Virtual displacement (or virtual work) method, 216
Virtual work, 213

Warping in torsion, 146
Welded joints, design of, 105
butt joints, 110
fillet welds, 105
Work, 89
units of, 18
virtual, 213
Yield conditions, 375
Yield point, 44
Yield strength, 44